On Some Transducer Synthesis Problems

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A basic zero-sum infinite game

Adam picks symbols in Σ

A basic zero-sum infinite game

Adam picks symbols in Σ

Eve picks symbols in Γ

A basic zero-sum infinite game

Adam picks symbols in Σ

Eve picks symbols in Γ

a

A basic zero-sum infinite game

Adam picks symbols in Σ

Eve picks symbols in Γ

aa

A basic zero-sum infinite game

Adam picks symbols in Σ

Eve picks symbols in Γ

aab

A basic zero-sum infinite game

Adam picks symbols in Σ

Eve picks symbols in Γ

$aabb$

A basic zero-sum infinite game

Adam picks symbols in Σ

Eve picks symbols in Γ

$aabbc$

A basic zero-sum infinite game

Adam picks symbols in Σ

Eve picks symbols in Γ

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Adam picks symbols in Σ

Eve picks symbols in Γ

$aabbcaa$

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$$aabbcaaabbaacb\dots$$

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$$u = aabbcaaabbaacb\dots$$

- Eve wins if $u \in W \subseteq (\Sigma\Gamma)^\omega$

Example: mimic Adam until first c

$$W = (aa + bb)^* c (\Gamma\Sigma)^\omega$$

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- Formally, she wins if she has a strategy $\lambda : \Sigma^+ \to \Gamma$ s.t.

$$\forall \sigma_1\sigma_2 \ldots \in \Sigma^\omega,\ \sigma_1\beta_1\sigma_2\beta_2 \cdots \in W$$

where $\beta_i = \lambda(\sigma_1 \ldots \sigma_i)$

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Example: Eve *always* mimics Adam

$$\lambda(u\sigma) = \sigma$$

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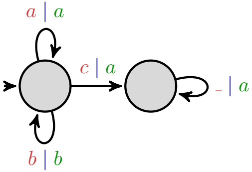

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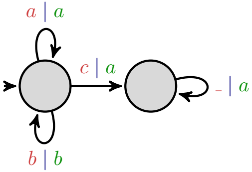

Complexity Results

- decidable problem [Büchi-Landweber 69] (via parity games)
- ExpTime-c from automata
- 2ExpTime-c for LTL specifications [Pnueli/Rosner 89]

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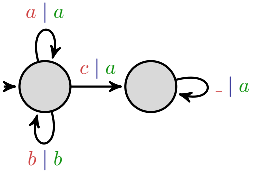

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- reactive synthesis competition SYNTCOMP since 2014

synchronous specification / implementation

Church synthesis as a uniformisation problem

Definition (Uniformiser)

Given $R \subseteq I \times O$, a uniformiser of R is a function $f : I \to O$ such that

1. for all $i \in dom(f)$, $(i, f(i)) \in R$,

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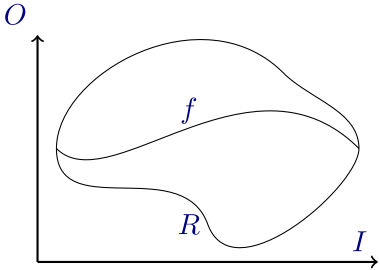

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Definition ($(\mathcal{S}, \mathcal{I})$-uniformisation problem)

Given a relation $R \in \mathcal{S}$, does there exists a uniformiser $f \in \mathcal{I}$?

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- $I = \Sigma^\omega$, $O = \Gamma^\omega$
- any $W \subseteq (\Sigma\Gamma)^\omega$ defines a relation $R_W \subseteq \Sigma^\omega \times \Gamma^\omega$:

$$R_W = \{(\sigma_1\sigma_2\ldots, \beta_1\beta_2\ldots) \mid \sigma_1\beta_1\sigma_2\beta_2\cdots \in W\}$$

AUT : class of ω-automatic relations.

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$$f_\lambda : \Sigma^\omega \to \Gamma^\omega$$

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R_W is uniformizable by f_λ iff λ is winning for Eve in the Church game with objective W.

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Reformulation of Büchi-Landweber's theorem

- The (AUT, STR)-uniformisation problem is decidable,
- and same as $(AUT, MEALY)$-uniformisation problem.

Wait and see: beyond Mealy machines

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Spec W: Replace each symbol by the following non $\perp$ symbol

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Talk Outline

1. beyond Mealy implementations
2. beyond automatic specifications

Beyond Mealy Implementations

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- based on works together with Vrunda Dave, Nathan Lhote, Krishna S. and Sarah Winter.

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- **Hypothesis**: specifications are automatic relations

Motivating question

Question

If a spec is not Mealy-realizable, is it still realizable in some larger class of implementations ?

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Theorem (Uniformization property)

Any automatic relation can be uniformized by an automatic function.

- f automatic if $\{\sigma_1\beta_1\sigma_2\beta_2\cdots \mid f(\sigma_1\sigma_2\ldots) = \beta_1\beta_2\ldots\}$ is ω-regular.
- in other words, any automatic spec is realizable by some automatic function
- result due to Siefkes 75 and Choffrut/Grigorieff 99, some proof in Carayol/Löding 2014 too

Proof of the theorem (for finite words)

Let $A = (Q, q_0, F, \Delta)$ be a DFA for R.

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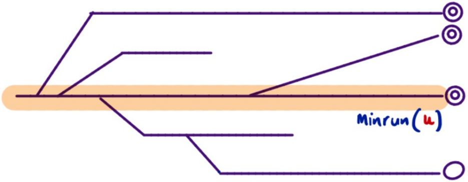

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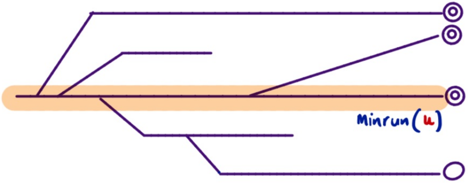

Lemma $L_{\min} = \{\text{minrun}(u) \mid u \in \Sigma^*\}$ is regular.

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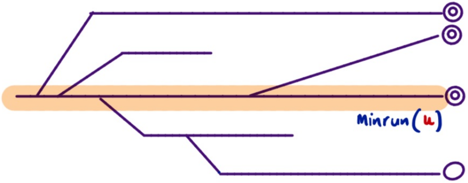

Lemma $L_{\min} = \{\text{minrun}(u) \mid u \in \Sigma^*\}$ is regular.

3. let $A_{\min}$ be a DFA accepting $L_{\min}$ and use it as a filter
4. the product $A \otimes A_{\min}$ defines an automatic function. $\square$

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- the latter theorem is not very useful for synthesis
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- f is automatic: $((\Sigma a)^* aa)^\omega + (\Sigma b)^* (bb)^\omega$
- f is not computable:
  no algorithm computes longer and longer output prefixes
  while reading longer and longer input prefixes

Turing-computability over infinite words

Consider a deterministic Turing machine M with 3 tapes:

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but $f_\perp \cup (\Diamond\Box\perp \mapsto \perp^\omega)$ is not.

Some characterization and some decidability result

- any computable function is continuous for the Cantor distance $d(u,v) = 0$ if $u = v$, and $2^{-|lcp(u,v)|}$ otherwise
- the converse is not true, but true in some cases:

Theorem (Dave,F.,Krishna,Lhote,19)

Let f be a function preserving regular languages under inverse image.

f is computable iff f is continuous.

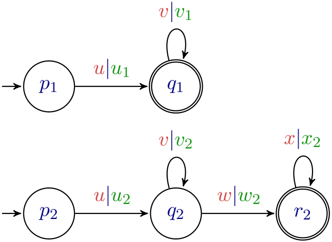
Theorem

Continuity (and so computability) is decidable in NLogSpace for automatic functions.

Proved for a large class (regular functions), already known for rational functions by Prieur,01.

Idea of the algorithm

1. Check the following forbidden pattern where $u_1 \neq u_2$:



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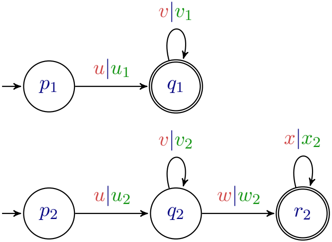

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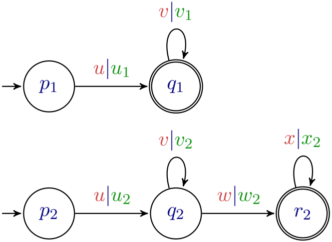

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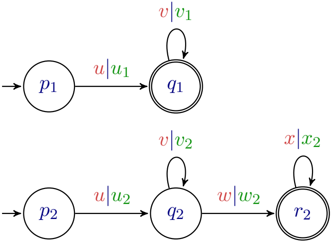

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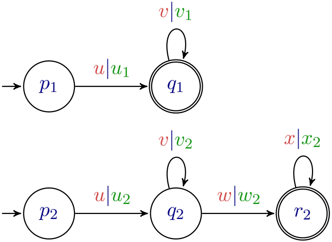

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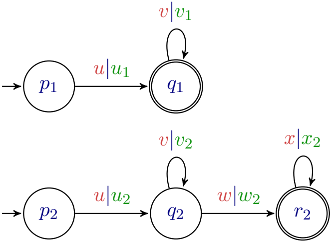

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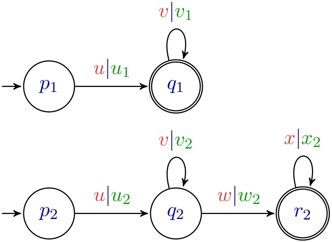

$$\lim_{n\to\infty} f(uv^n wx^\omega) \neq f(\lim_{n\to\infty} uv^n wx^\omega)$$

$$\lim_{n\to\infty} f(uv^n wx^\omega) = \lim_{n\to\infty} u_2 v_2^n w_2 x_2^\omega = u_2 v_2^\omega$$

$$f(\lim_{n\to\infty} uv^n wx^\omega) = f(uv^\omega)$$

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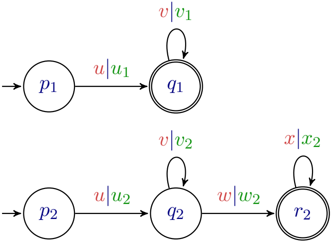

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Back to synthesis

Problem

Input: an automatic specification $R \subseteq \Sigma^\omega \times \Gamma^\omega$

Output: Is R uniformisable by some computable function f ?

$f \subseteq R$ and $\mathrm{dom}(f) = \mathrm{dom}(R)$

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- 2EXPTIME when R has **total domain** and is given as a DPA Holtmann/Kaiser/Thomas'10
- EXPTIME-c Klein/Zimmermann'14
- EXPTIME-c even when R has **partial domain** F., Winter,21

Partial vs Total Domain

- classical formulation of Church synthesis

$$\exists\lambda_{Eve} : \Sigma^* \to \Gamma \cdot \forall\sigma_1\sigma_2\dots \in \Sigma^\omega \cdot (\sigma_1\sigma_2\dots, \lambda(\sigma_1)\lambda(\sigma_1\sigma_2\dots)) \in R$$

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From partial to total domain

- $R \mapsto R_{tot} = R \cup \overline{\mathrm{dom}(R)} \times \Gamma^\omega$
- R is realizable under assumption $\mathrm{dom}(R)$ iff R_{tot} is realizable (in the classical sense)
- R is automatic iff R_{tot} is automatic

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Partial vs Total Domain

- the latter reduction fails if one asks realizability by a computable function instead of a strategy function
- it does not preserve the existence of a computable realizer

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Counter-example

- $R = f_\perp$:

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- R is uniformizable by a computable function ($f_\perp$ **is** computable)
- R_{tot} is not

Transducer synthesis (total case)

Theorem (Holtmann/Kaiser/Thomas'10, Klein/Zimmermann'14)

*Let $R \in AUT$ with **total** domain.*
If R is uniformisable by a computable function, it is uniformisable by a deterministic (one-way) transducer.

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- R_1: behaves as $f_\perp$ as long as there is no two consecutive $\perp$, otherwise output $\perp$ forever.

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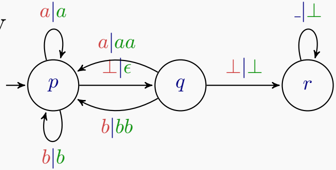
- R_1 is a function of total domain
- it is computed by



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- total setting: Eve only needs to wait a constant number of steps
- partial setting: she may need to wait arbitrarily long

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Theorem (F., Winter)

*Let $R \in AUT$ with **partial** domain.*
*If R is uniformisable by a computable function, it is uniformisable by a deterministic **two**-way transducer.*

Two-wayness is necessary:

Uniformisation by computable functions: proof idea

- reduction to a turn-based two-player parity game where
- Adam plays input letters
- Eve plays output letters or a waiting action

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Solution

- store state transformations of the specification automaton induced by Adam's inputs
- monitor membership to the domain
- Eve picks a state transformation instead of concrete outputs
- winning condition: if Adam's input is in the domain, then Eve infinitely often picks an accepting state transformation
- convert this into a parity condition

Summary of Part 1

- synthesis of computable functions from automatic specifications given by DPA is ExpTime-c
- without assuming that inputs comes from the domain: deterministic one-way transducer suffice
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Can we go beyond automatic specifications ?

Beyond Automatic Specifications

Automata model for asynchronous spec / impl

- Non-automatic spec: alphabet $\Sigma = \{0, +, -\}$, $\Gamma = \{0, \#\}$.

$$R \ : \ +0^{n_1} - 0^{n_2} + 0^{n_3} \ldots \mapsto \#0^{n_1+1}\#0^{n_2-1}\#0^{n_3+1} \ldots$$

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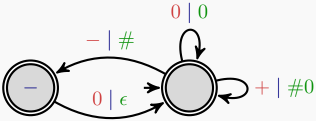

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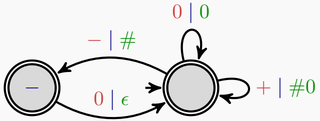
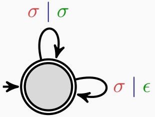


- non-deterministic, define rational relations (RAT):



Undecidability Result

Theorem (Löding,Carayol, 14)

The uniformization problem of rational relations by sequential functions is undecidable.

Proof Overview

Post-correspondence problem (PCP)

Given $(u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n) \in \{0,1\}^* \times \{0,1\}^*$, find indices $i_1, \ldots, i_k$ such that

$$u_{i_1} \ldots u_{i_k} = v_{i_1} \ldots v_{i_k}.$$

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Reduction: $\Sigma = \{1, \ldots, n, \#, A, B\}$, $\Gamma = \{0, 1, \#, A, B\}$.

Construct R such that:

$$i_1 \ldots i_k \# \alpha \mapsto \begin{cases} u_{i_1} \ldots u_{i_k} \#^\omega & \text{if } |u|_a = \infty \\ v\#^\omega,\ v \in \{0,1\}^* \setminus \{v_{i_1} \ldots v_{i_k}\} & \text{otherwise.} \end{cases}$$

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Correctness

- no PCP solution $\Rightarrow$ always output $u_{i_1} \ldots u_{i_j} \#^\omega$
- $i_1, \ldots, i_k$ is a PCP solution: reading $i_1 \ldots i_k \#$, any uniformiser can decide what to output w/o reading the infinite suffix

Asynchronous specifications: hopeless ?

- what about computable functions ?

[2] $\exists k \forall u \in \Sigma^\omega |\{v \mid (u,v) \in R\}| \leq k$

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Asynchronous specifications: hopeless ?

- what about computable functions ? still undecidable
- what about finite words and sequential functions ? still undecidable
- decidability recovered for finite-valued rational relations of finite words[F.,Jecker,Löding,Winter,16] [2]
- and for stronger inclusion notions $f \subseteq R$ [F.,Jecker,Löding,Winter,16]

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A logic for asynchronous specifications

- so far, specifications are given by automata or transducers
- what about a logic ?

A logic for asynchronous specifications

- so far, specifications are given by automata or transducers
- what about a logic ?
- for automatic specifications: LTL, MSO, well-known

Goal

Define a logic which is

- expressive enough to capture a large class of asynchronous specifications
- has decidable satisfiability problem
- have decidable model-checking problem for known transducer models
- have some decidable synthesis problems

A logic for asynchronous specifications

- so far, specifications are given by automata or transducers
- what about a logic ?
- for automatic specifications: LTL, MSO, well-known

Goal

Define a logic which is

- expressive enough to capture a large class of asynchronous specifications
- has decidable satisfiability problem
- have decidable model-checking problem for known transducer models
- have some decidable synthesis problems

We introduce a logic to define finite word relations.

Motivation: Model-checking and Synthesis

- model-checking automata: $A \models \varphi$ if $L_A \subseteq L_\varphi$

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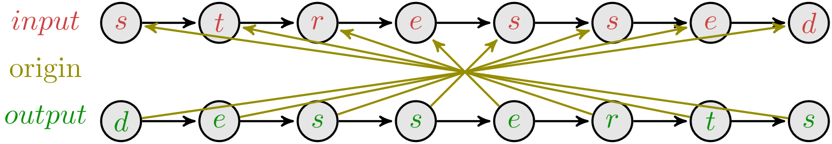
Every task is scheduled exactly once	$\{(t_1 \dots t_n, t_{\pi(1)} \dots t_{\pi(n)}) \mid \pi$ is a permutation $\}$

MSO over Origin Graphs (MSO$_o$)

- see pairs (u, v) as structures with origin information (origin graphs)



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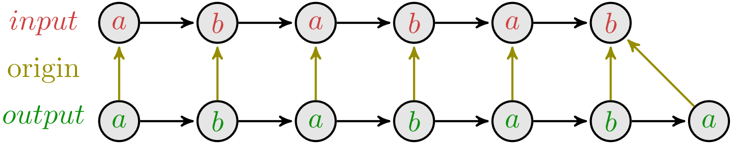

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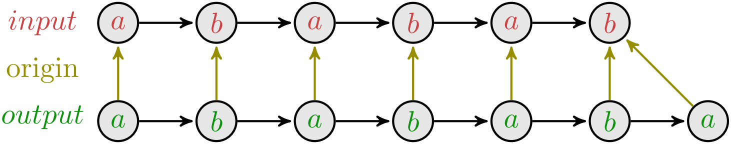
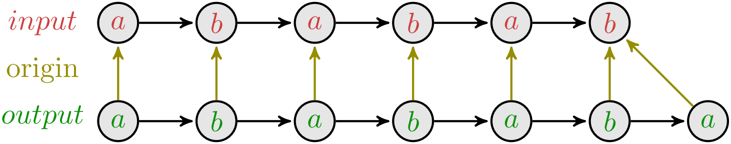


- MSO$_o$:= MSO$[\leq_{in}, \leq_{out}, o]$

MSO over Origin Graphs (MSO_o)

- see pairs (u, v) as structures with origin information (origin graphs)



- $\text{MSO}_o := \text{MSO}[\leq_{in}, \leq_{out}, o]$
- a formula φ defines a set of origin graphs $\{g \mid g \models \varphi\}$
- and hence a relation (by projecting away the origin)

Examples

- True

$$\top$$

Examples

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$$\top$$

- There is an a in the output

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- Mirror

$$\phi_s \wedge \forall^{out} x, y\ x \leq_{out} y \rightarrow o(y) \leq_{in} o(z)$$

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Mikolaj Bojańczyk's Observation (14) Most transducer models T define a set of origin graphs $ographs(T)$

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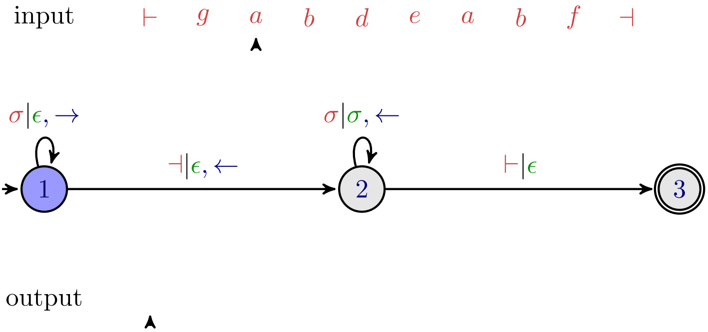

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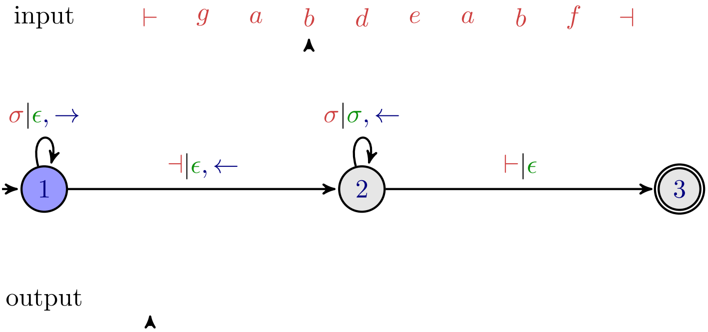

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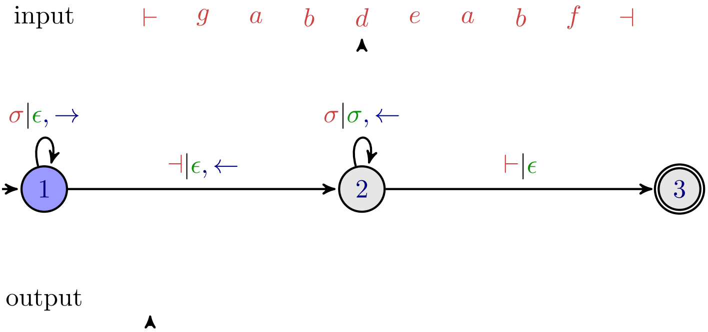

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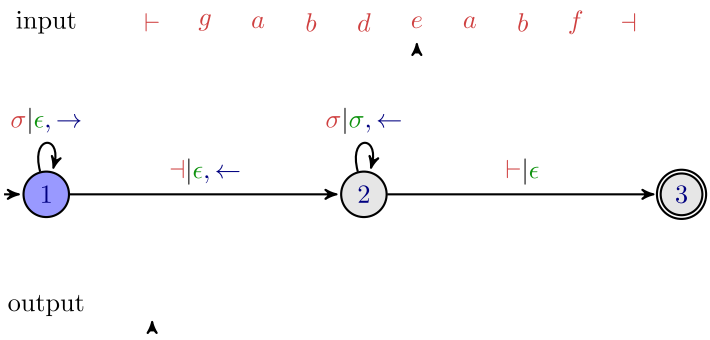

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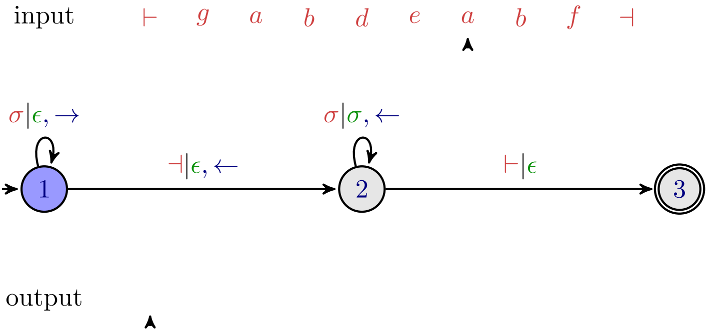

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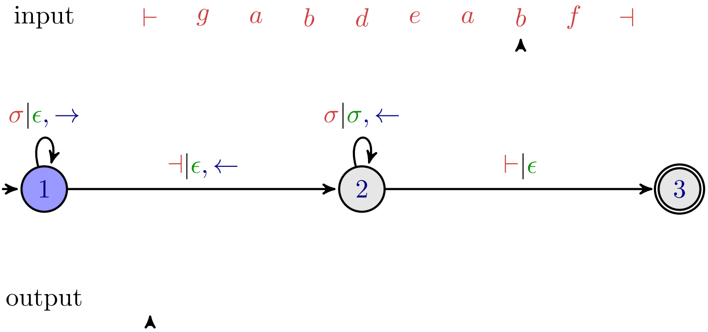

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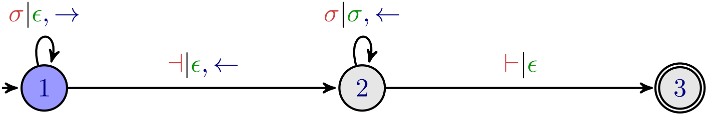
input $\vdash$ g a b d e a b f $\dashv$



output

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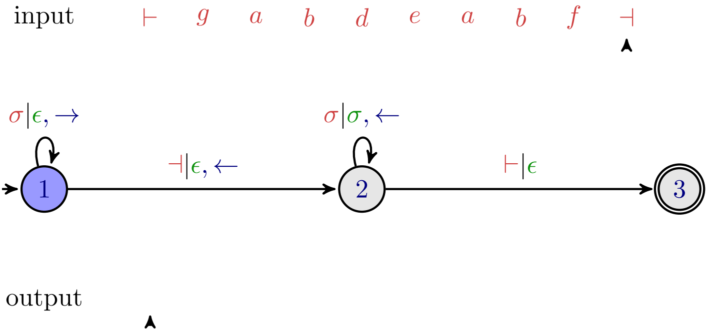

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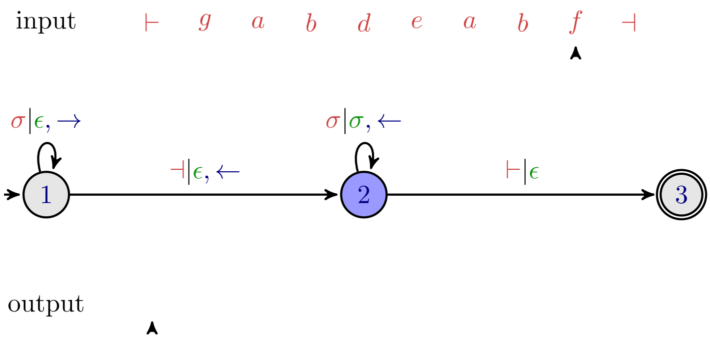

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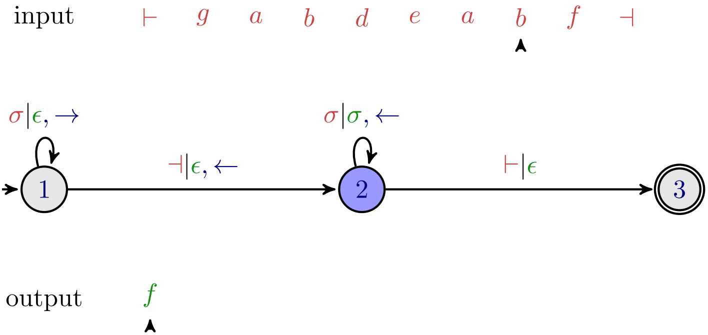

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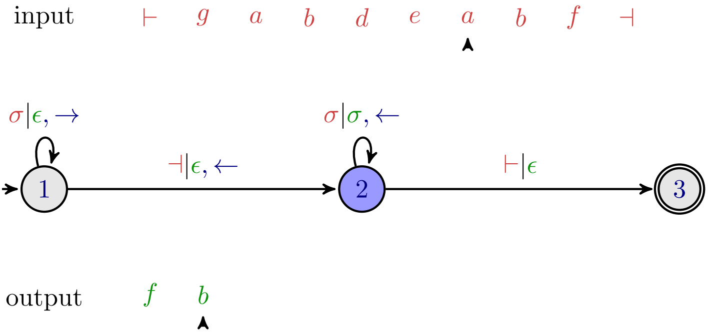

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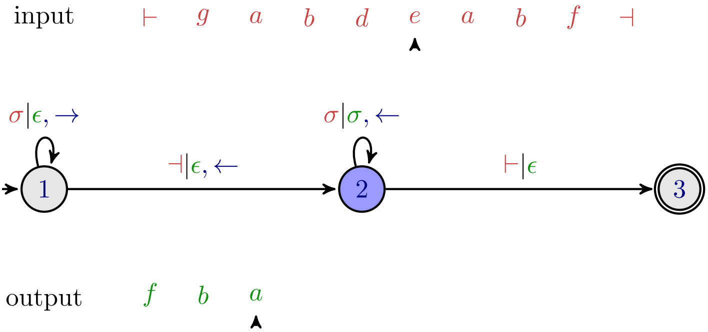

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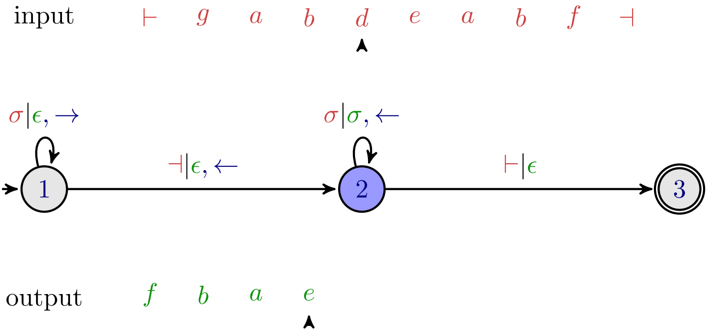

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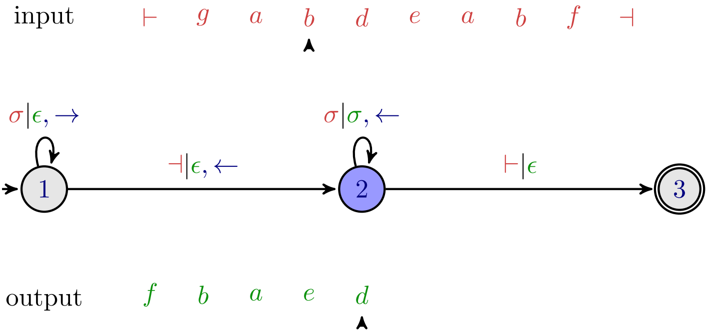

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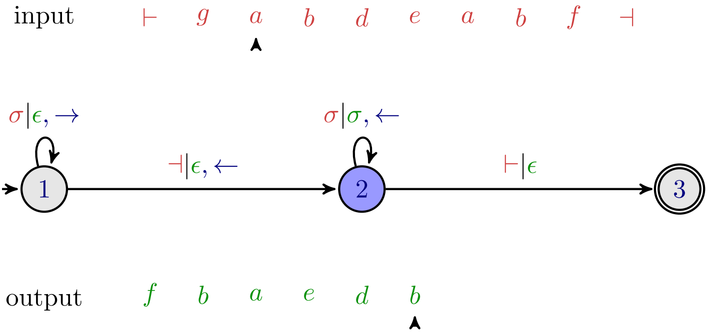

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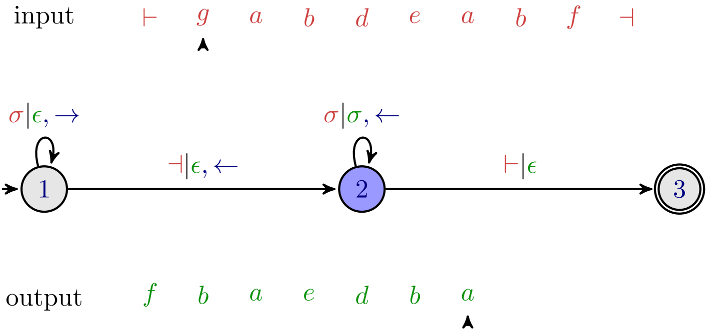

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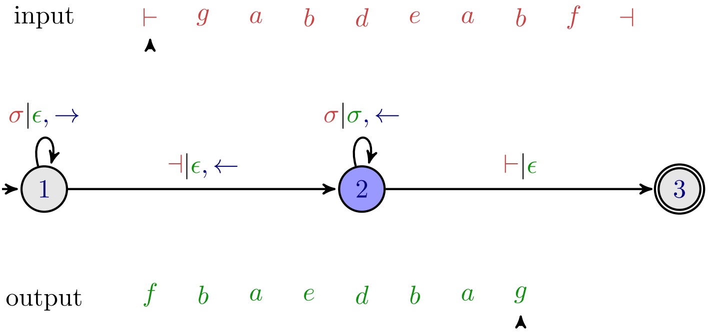

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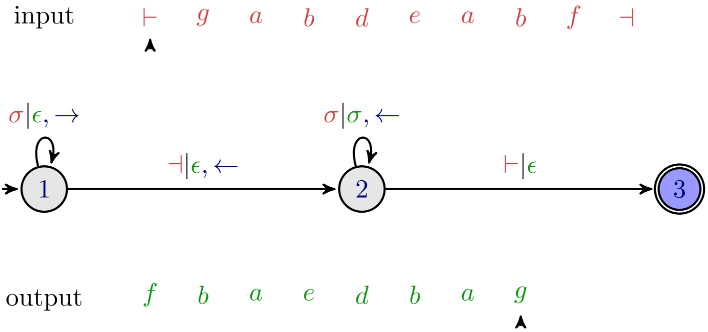

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Other examples of two-way transductions

- doubling function $aabbc \mapsto aabbcaabbc$
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Many Characterisations

- reversible 2-way transducers Dartois, Fournier, Jecker, Lhote, 17
- deterministic 1-way transducers with registers Alur, Cerny, 09
- MSO-transductions on strings Engelfriet,Hoogeboom,01
- regular combinators (Alur, Freilich, Raghothaman, 14) (Dave, Gastin,

Results

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Theorem (Bojanczyk,Daviaud,Guillon,Penelle,17)

The model-checking problem $ographs(T) \models \varphi$ *for* T *deterministic 2-way trans and* $\varphi \in MSO_o$ *is decidable.*

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Contribution Dartois,F.,Lhote,18

A fragment $\mathcal{F} = FO_2[\leq_{out}, MSO_{bin}[\leq_{in}]]$such that:

1. any $\mathcal{F}$-defined relation is uniformisable by a det. 2-way transducer
2. satisfiability is decidable
3. it is expressive (captures regular functions and more)
4. implies decidability of an expressive logic for data words: $FO_2[\leq_{pos}, MSO_{bin}[\leq_{data}]]$

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beyond Mealy machines

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Conclusion

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- for instance: synthesis of computable functions from synchronous specifications given as weighted automata ?
- other interesting question: synthesis under assumptions (regular, quantitative, etc.)