# An Introduction to Petri nets and how to analyse them...

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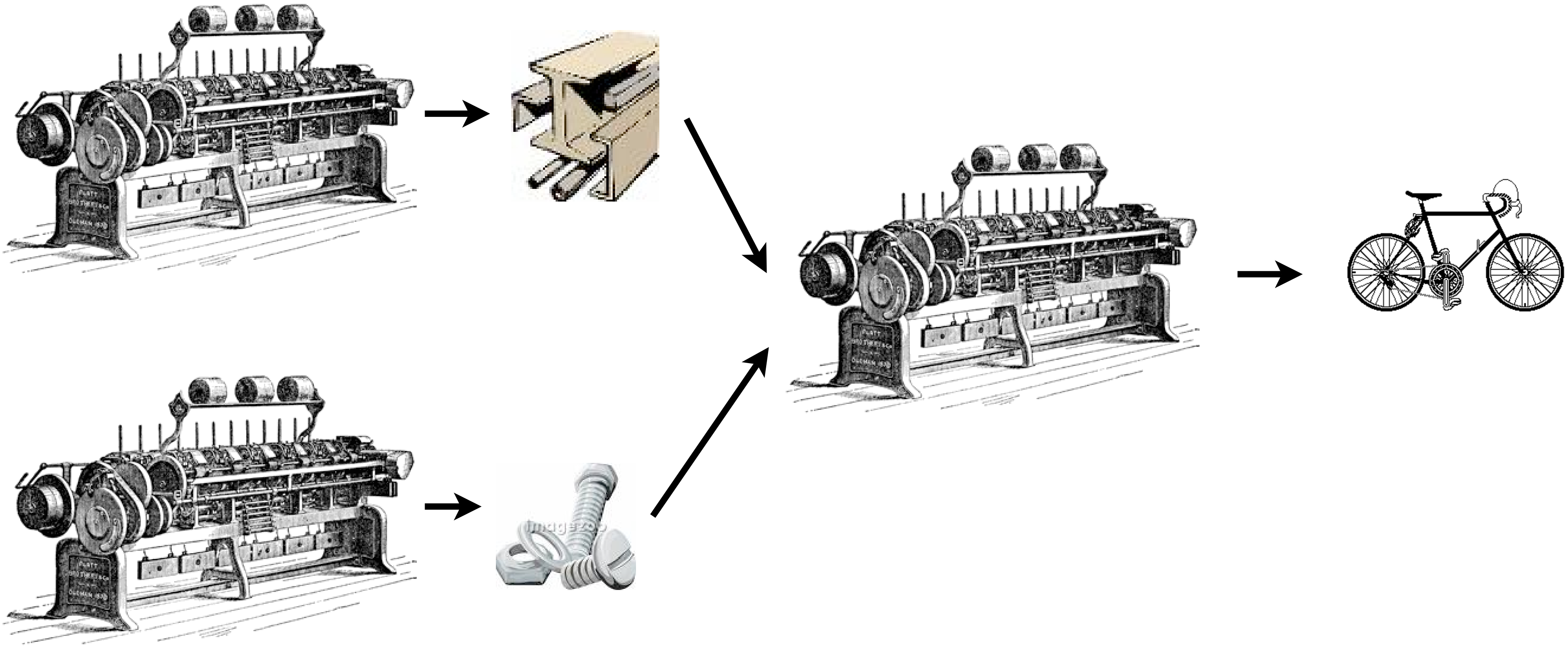
# Introduction

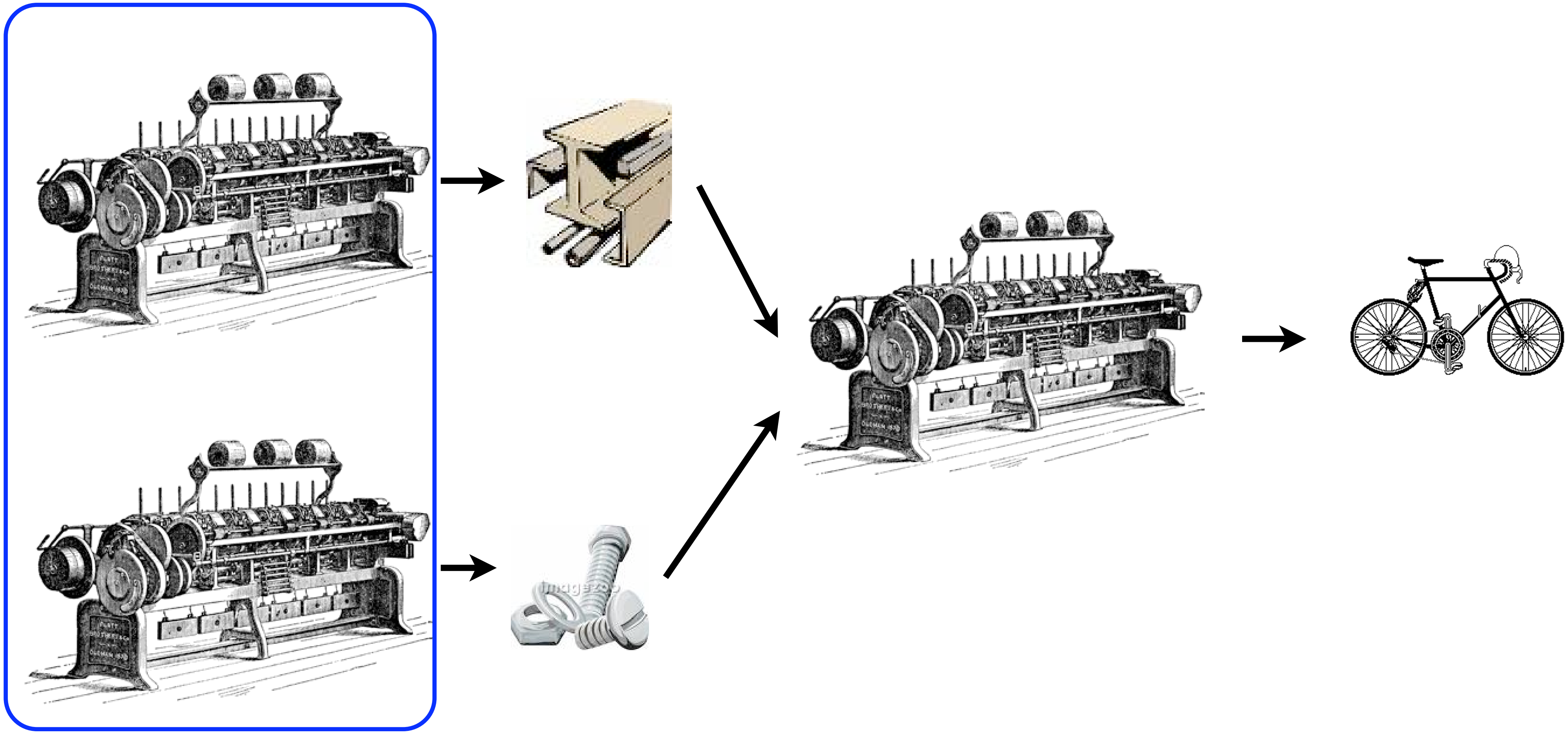
# Introduction

- Concurrency: property of a "system" in which many "entities" act at the same time and interact.
  - Often found in many application:
    - Computer science (e.g.: parallel computing)
    - Workflow
    - Manufacturing systems
    - ....

# Introduction
# Concurrency

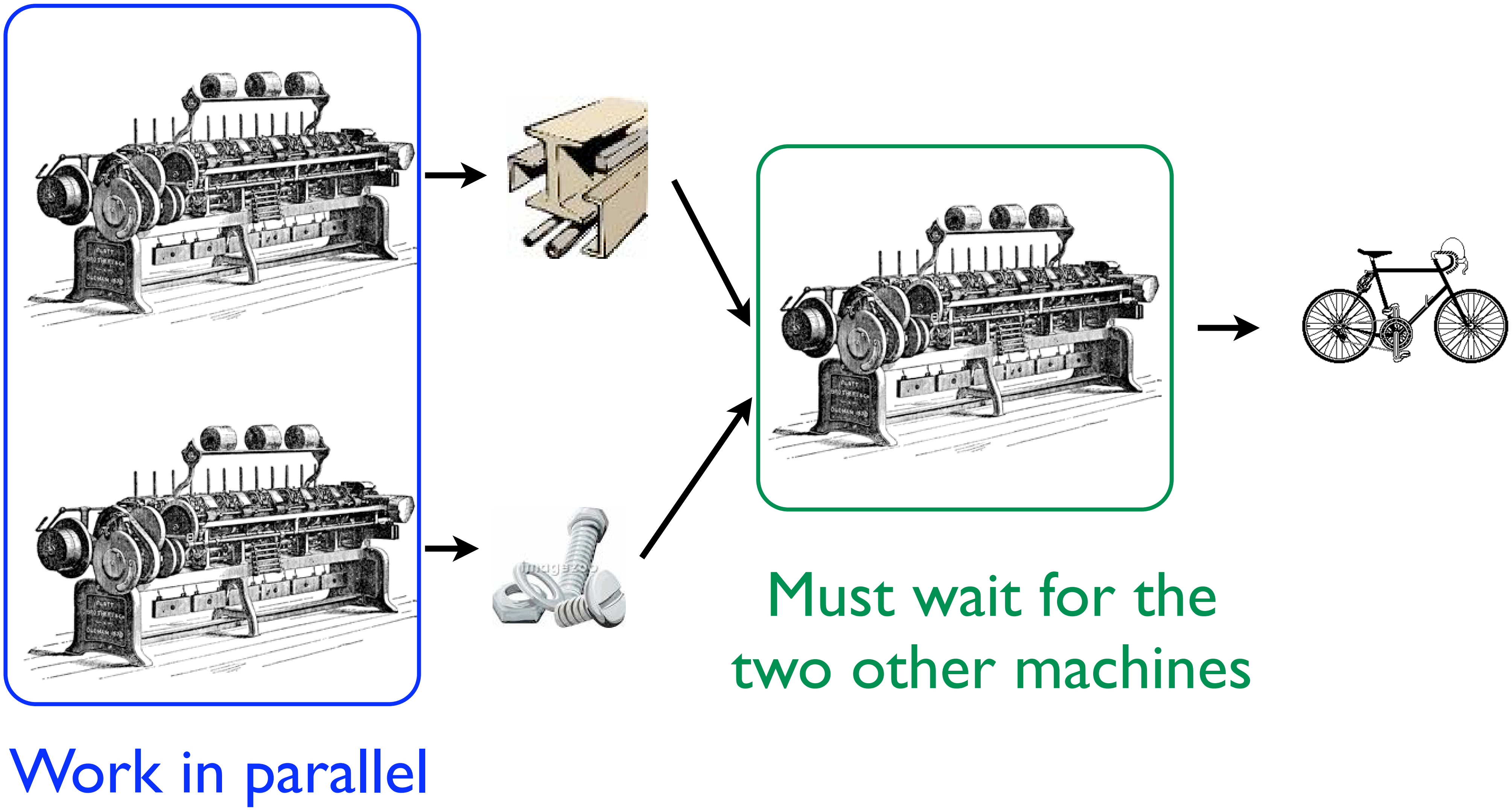


# Introduction
# Concurrency

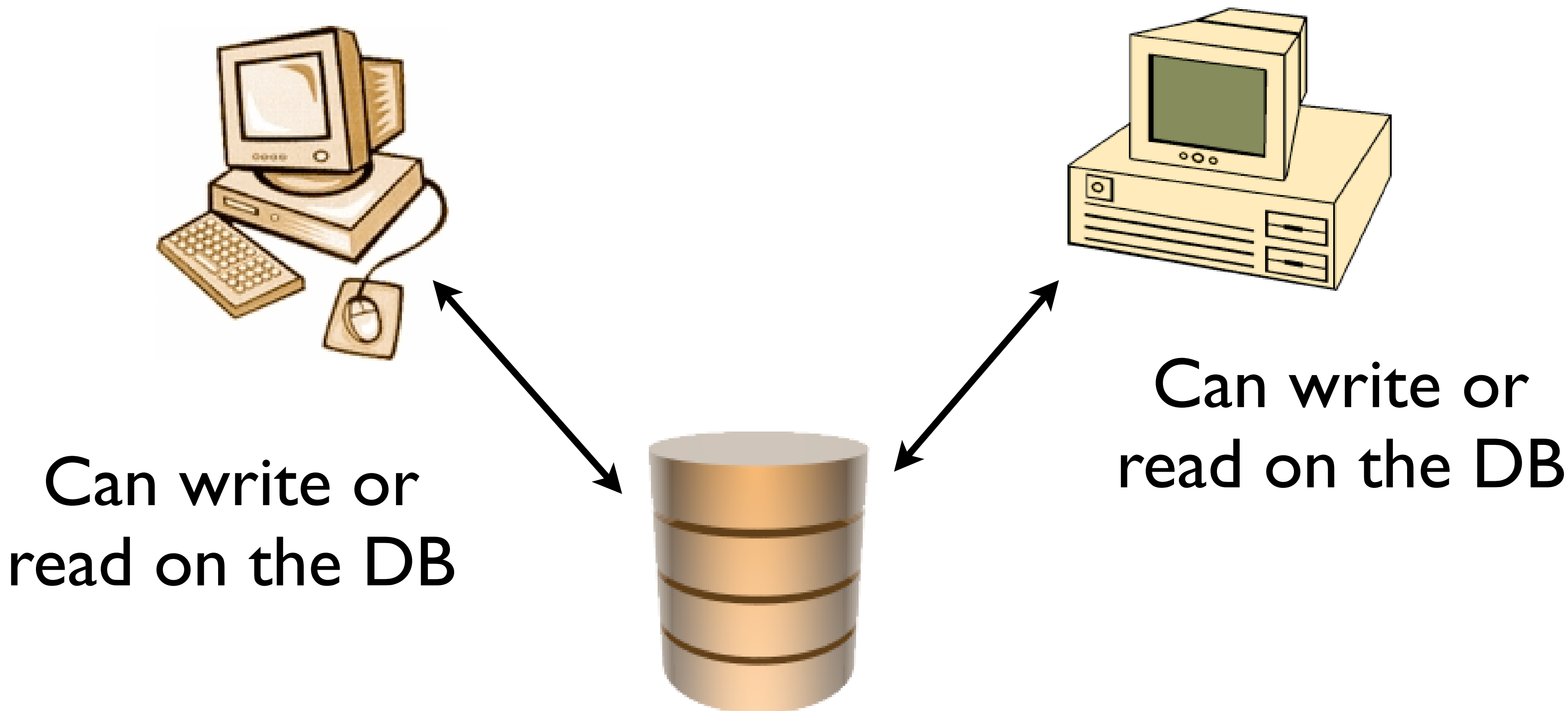


Work in parallel

# Introduction
# Concurrency



# Introduction
# Concurrency



# Introduction
# Concurrency

Boss

# Introduction
# Concurrency

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# Introduction
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Employees: work in parallel

# Introduction
# Concurrency

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gives work

Employees: work in parallel

# Introduction
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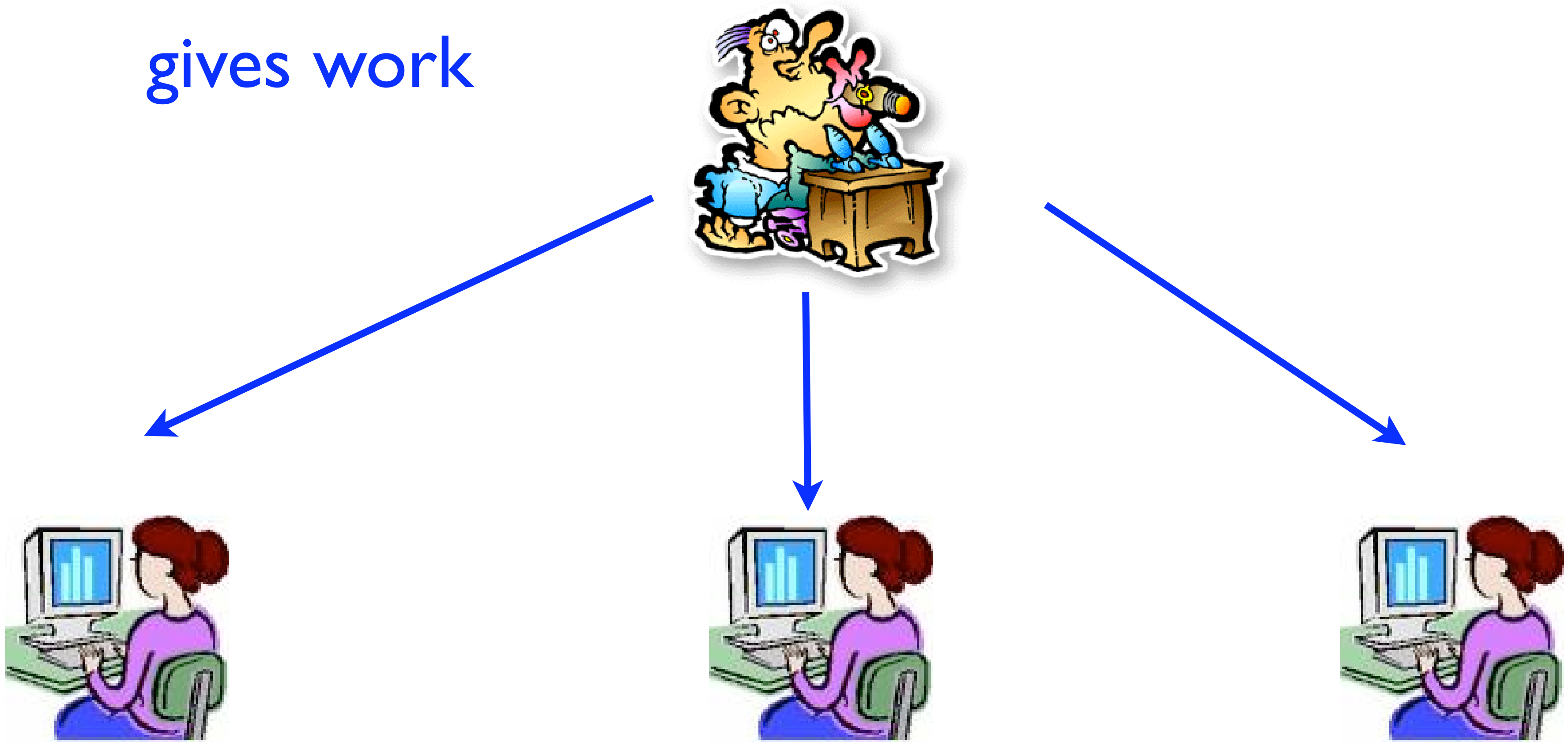


Employees: work in parallel

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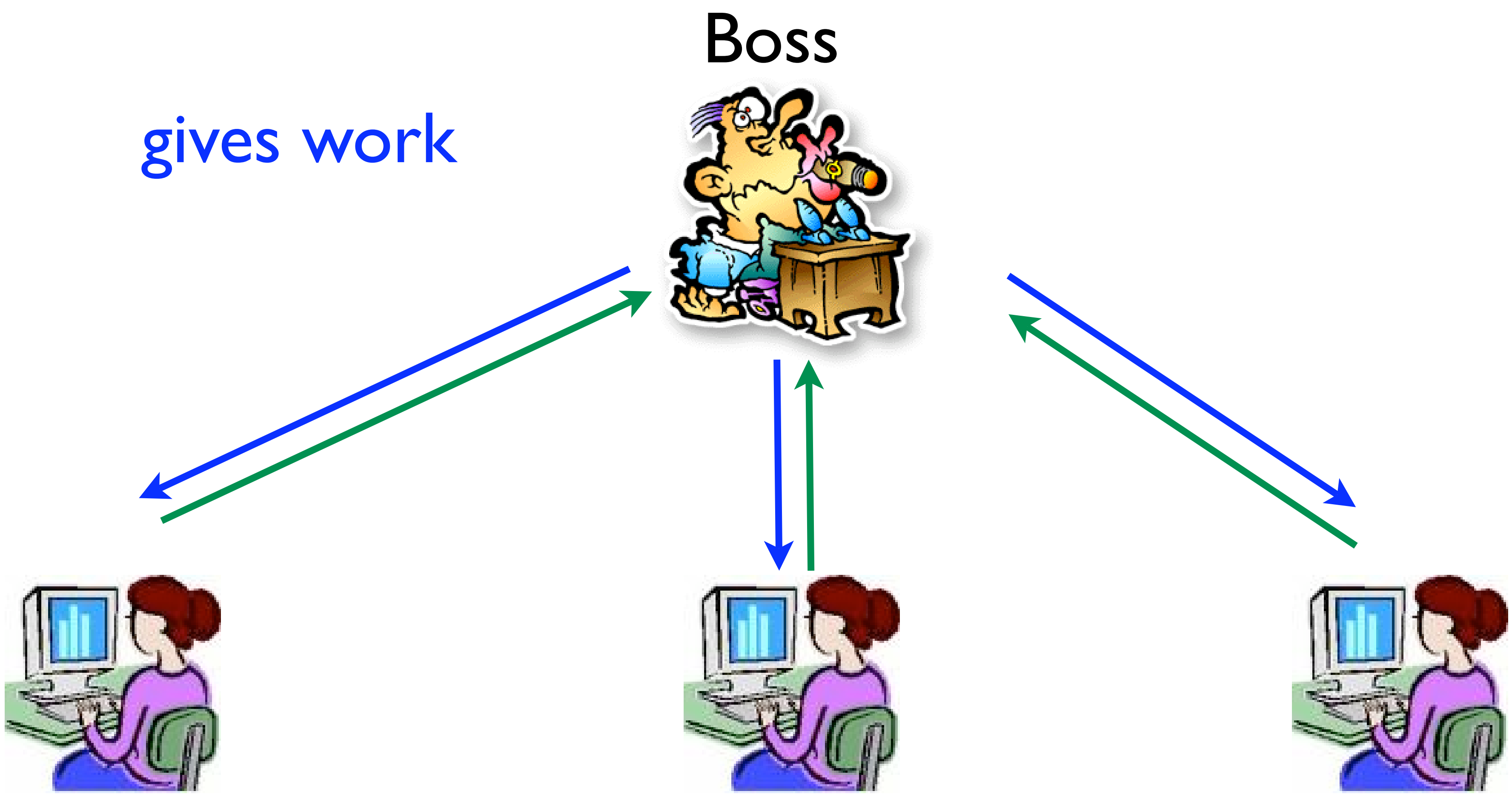
# Concurrency




Employees: work in parallel

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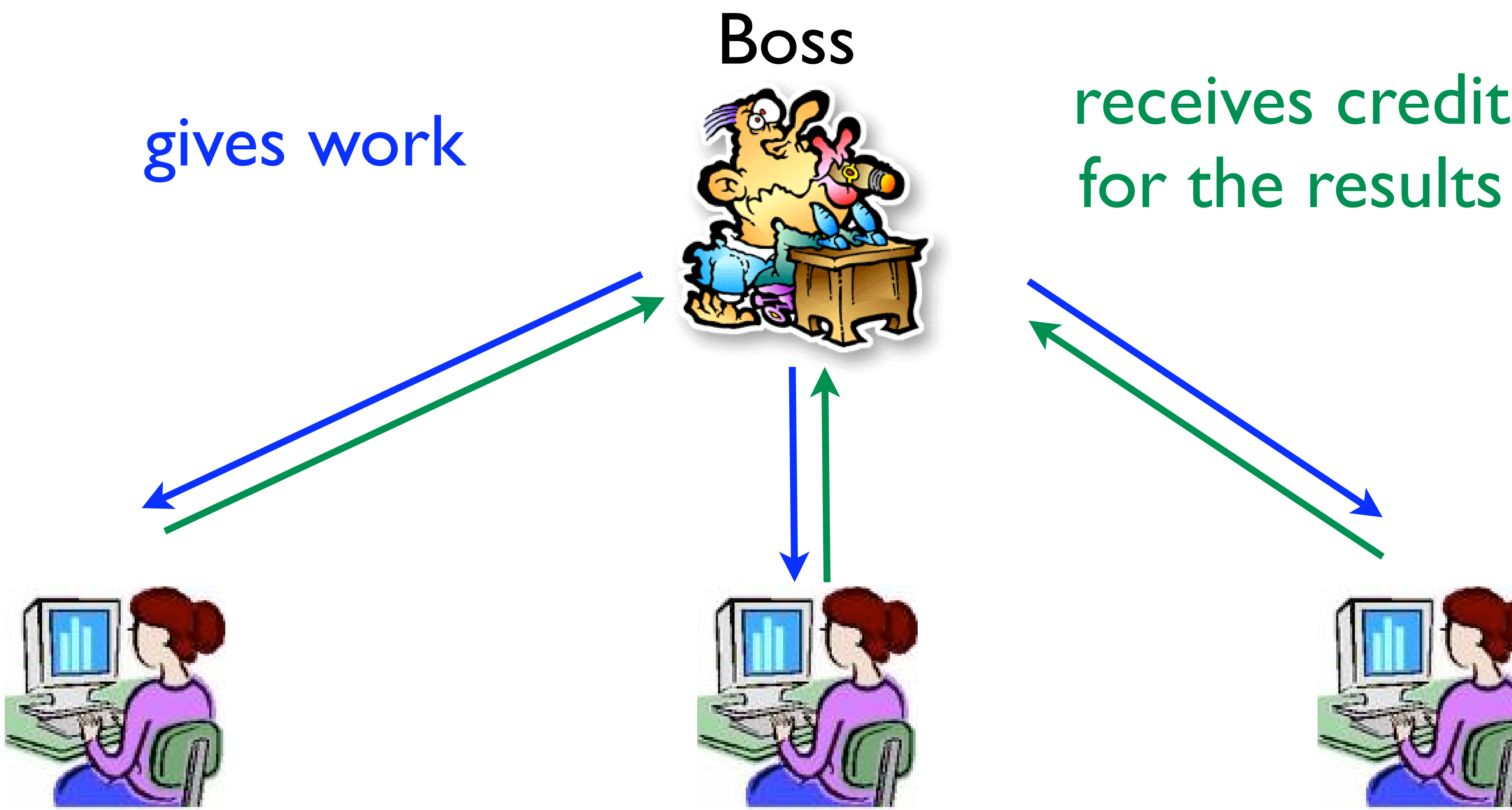



Employees: work in parallel

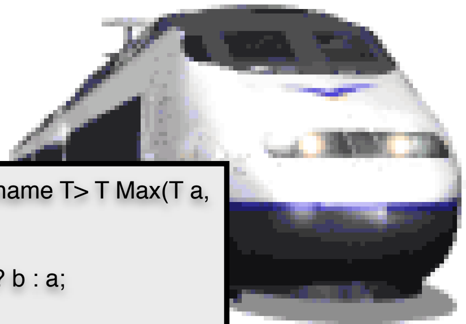
# Introduction

- Petri nets are a tool to model concurrent systems and reason about them.
- Invented in 1962 by C.A. Petri.

# The aim of the talk

- Introduce you to Petri nets (and some of their extensions)
- Explain several analysis methods for PN
  - i.e., what can you ‘ask’ about a PN ?
- Give a rough idea of the research in the verification group at ULB...
  - ... and foster new collaborations ?

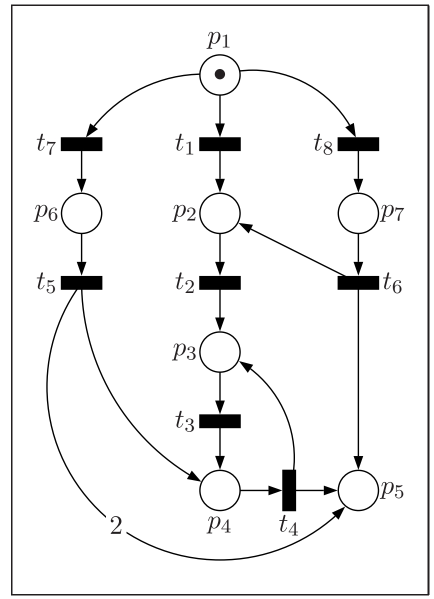
# How I use Petri nets



```
template <typename T> T Max(T a,
T b)
{
   return a < b ? b : a;
}

#include <string>
int main()  // fonction main
{
   int i = Max(3, 5);
   char c = Max('e', 'b');
   std::string s = Max(std::string
("hello"), std::string("world"));
   float f = Max<float>(1, 2.2f);
```

abstraction

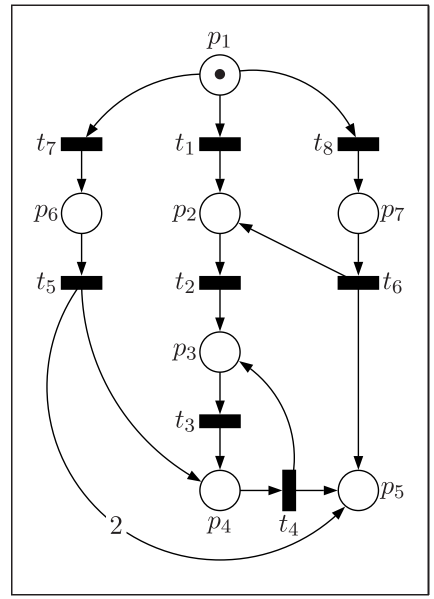



Analysis method
of PN

# How you might use PN

Your favorite application

abstraction

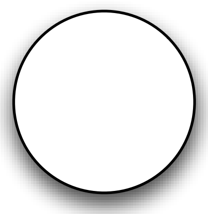


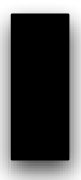

Analysis method of PN

# Intuitions

# Ingredients

A Petri net is made up of...

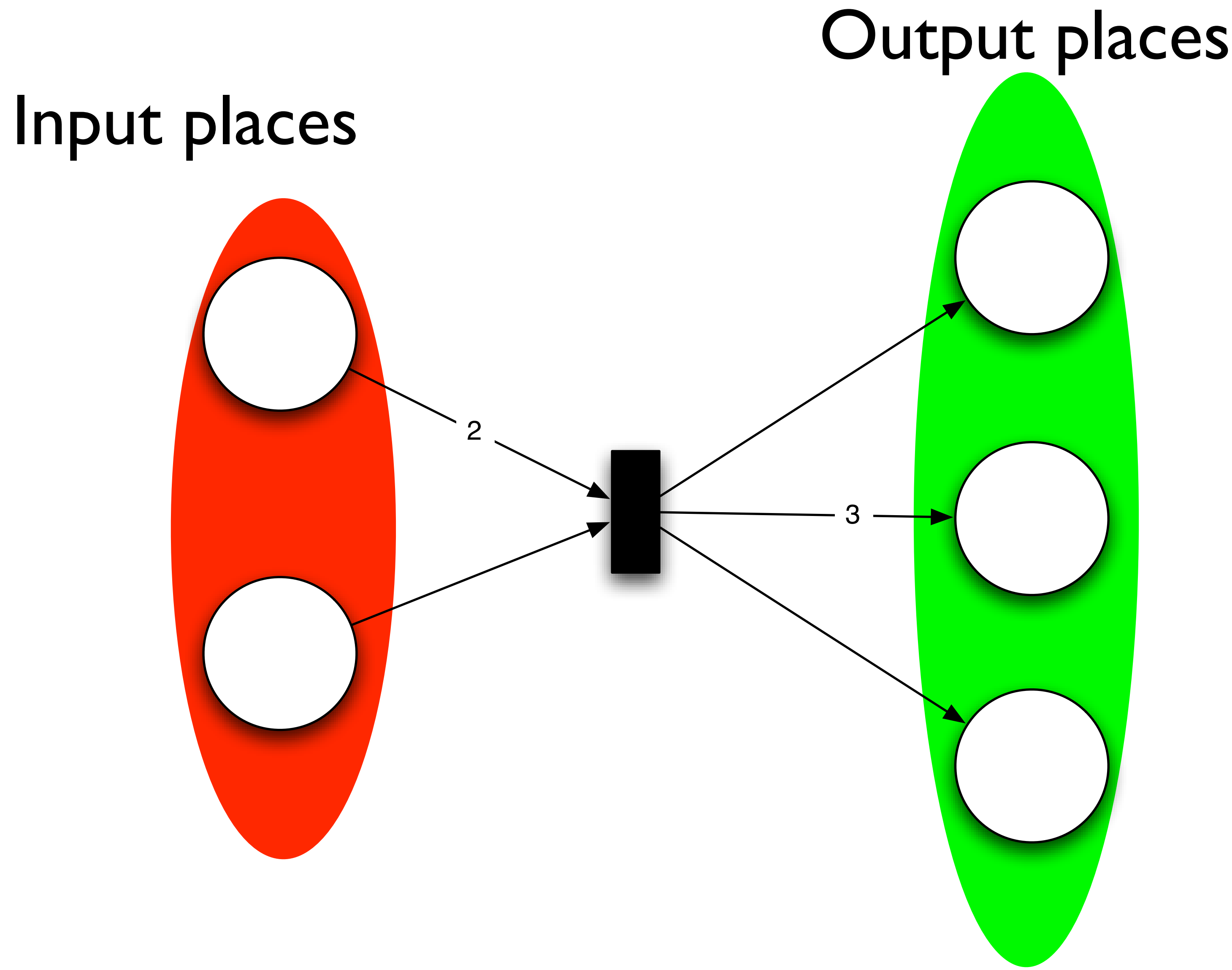
Places  = some type of resource

Transitions  consume and produce resources

Tokens  = one unity of a certain resource

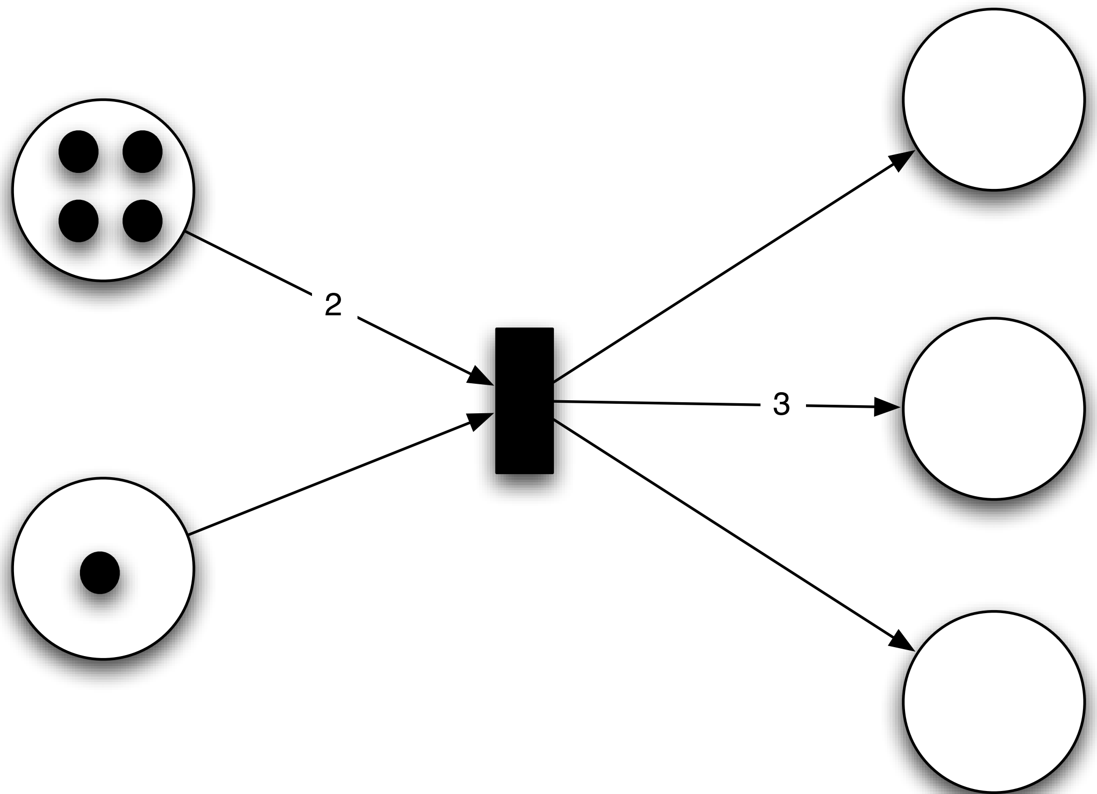
Tokens 'live' in the places

# Transitions



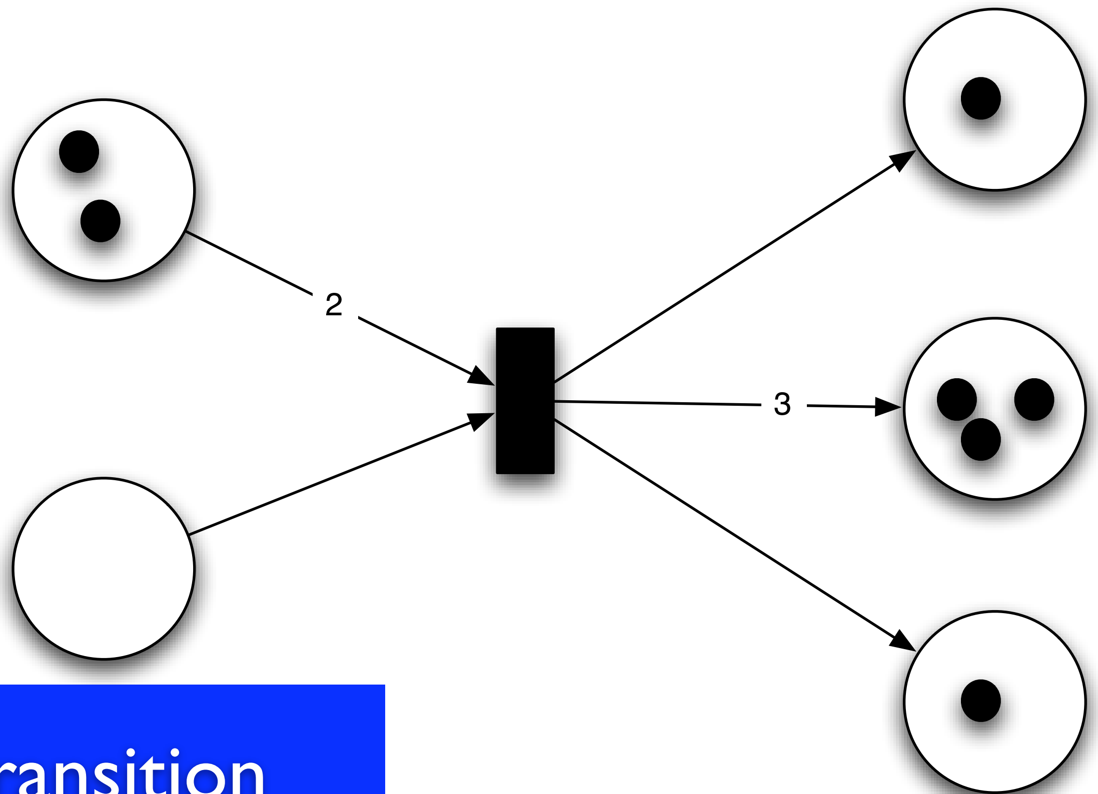
# Firing a transition

Transitions consume tokens from the input places and produce tokens in the output places



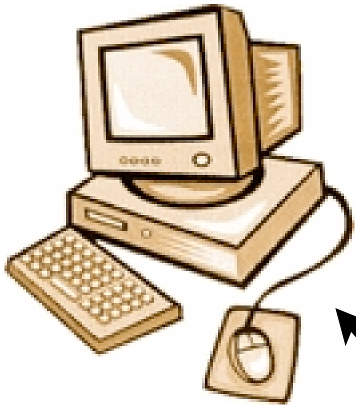
# Firing a transition

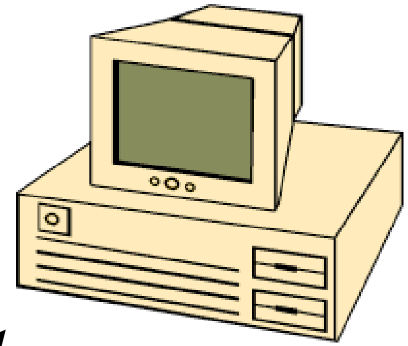
Transitions consume tokens from the input places
and produce tokens in the output places




Now, the transition
cannot be fired anymore

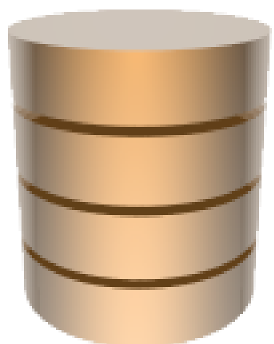
# Example 1





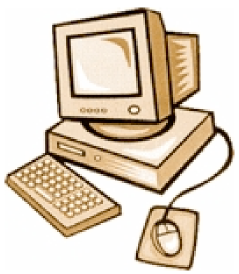
Can write or read on the DB

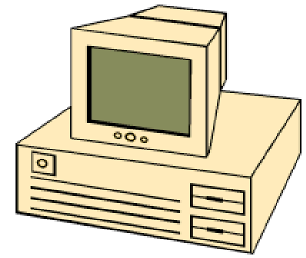
Can write or read on the DB

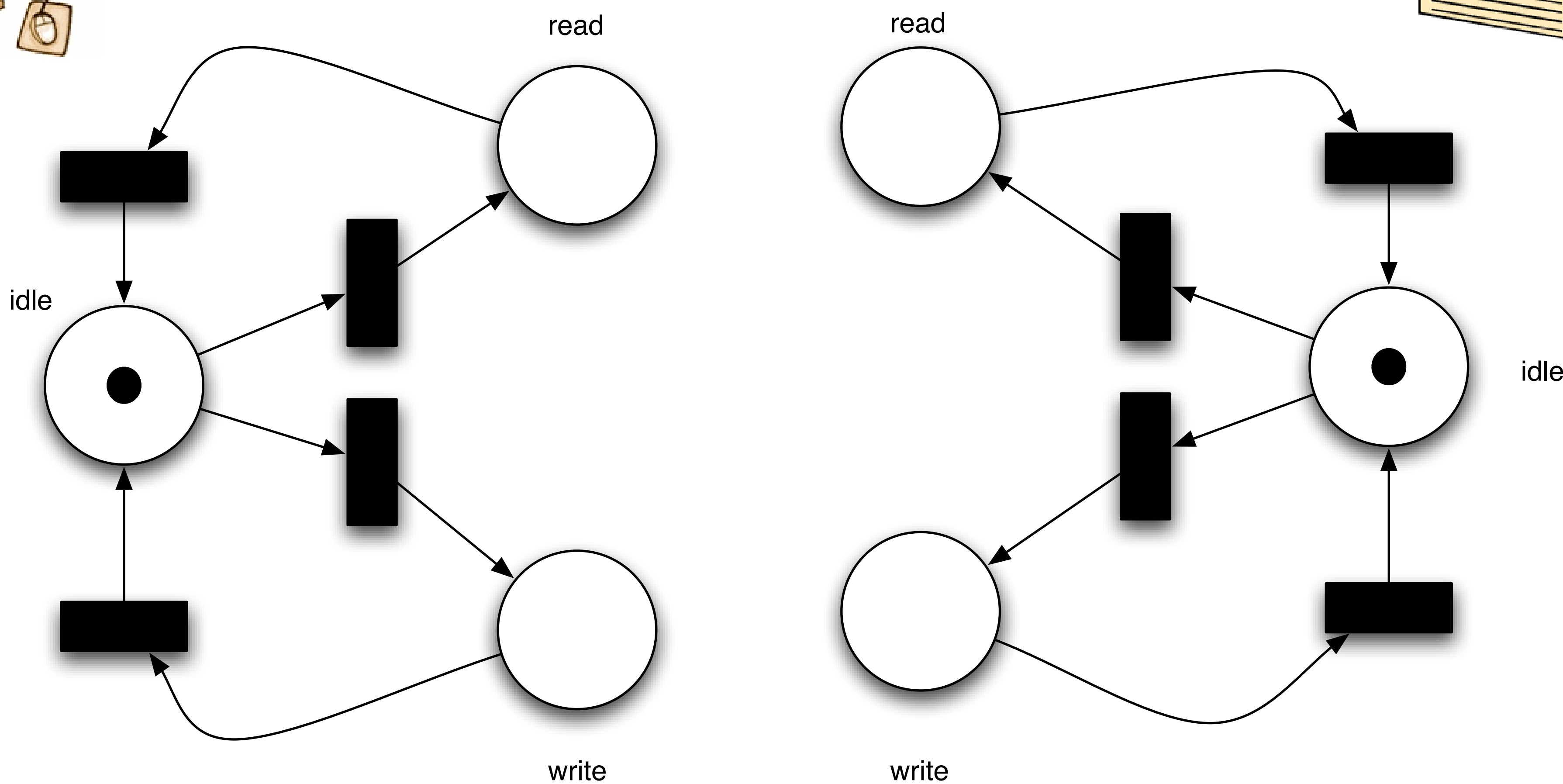


The two machines cannot write at the same time

# Example 1

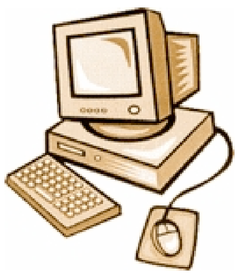


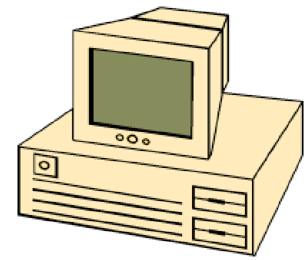


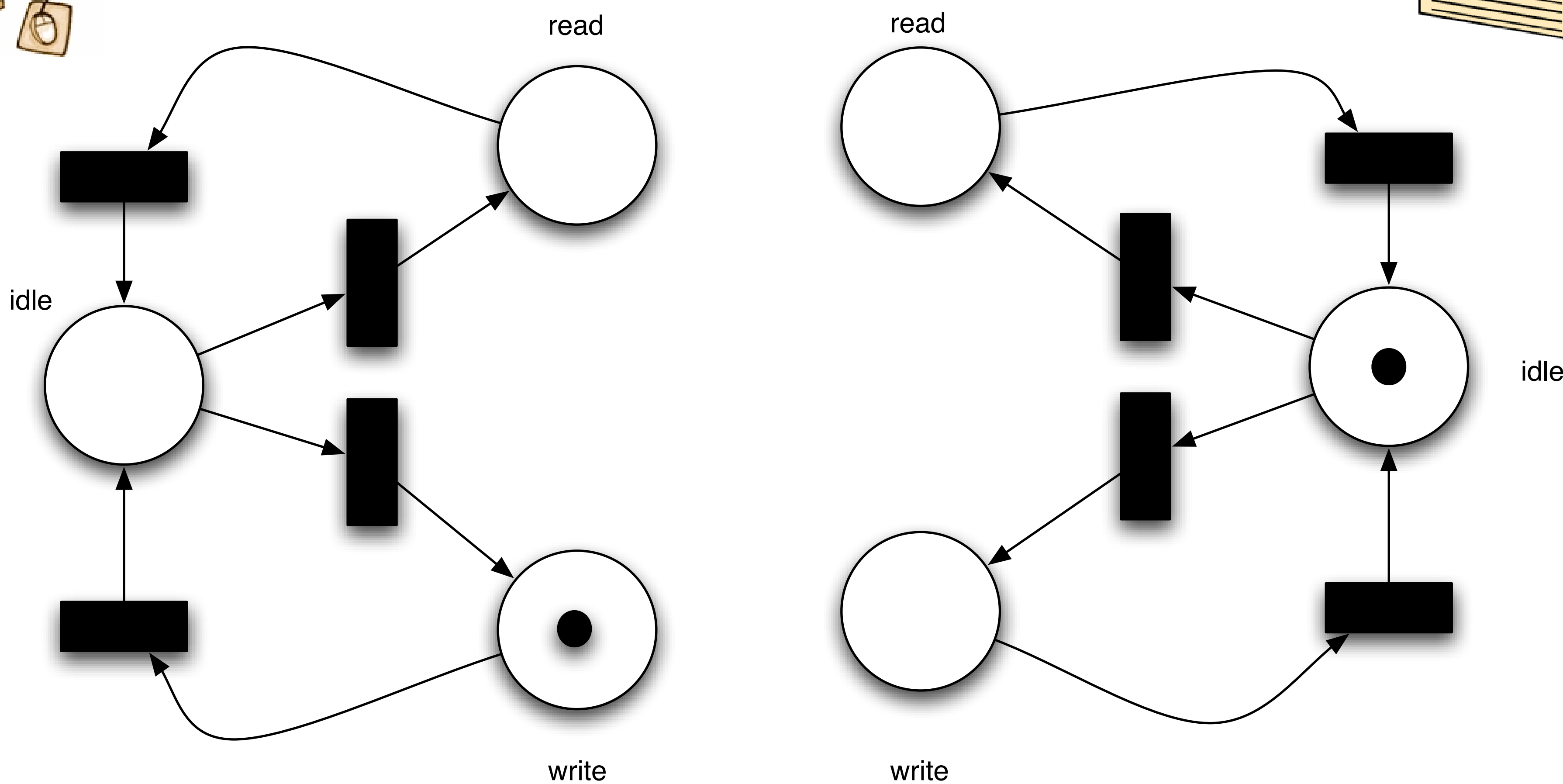



The token tells us the state of the process

# Example 1





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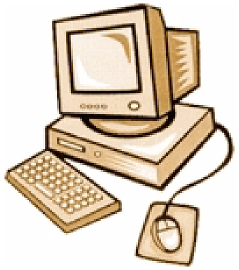

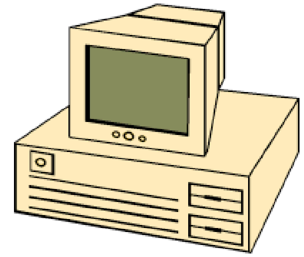

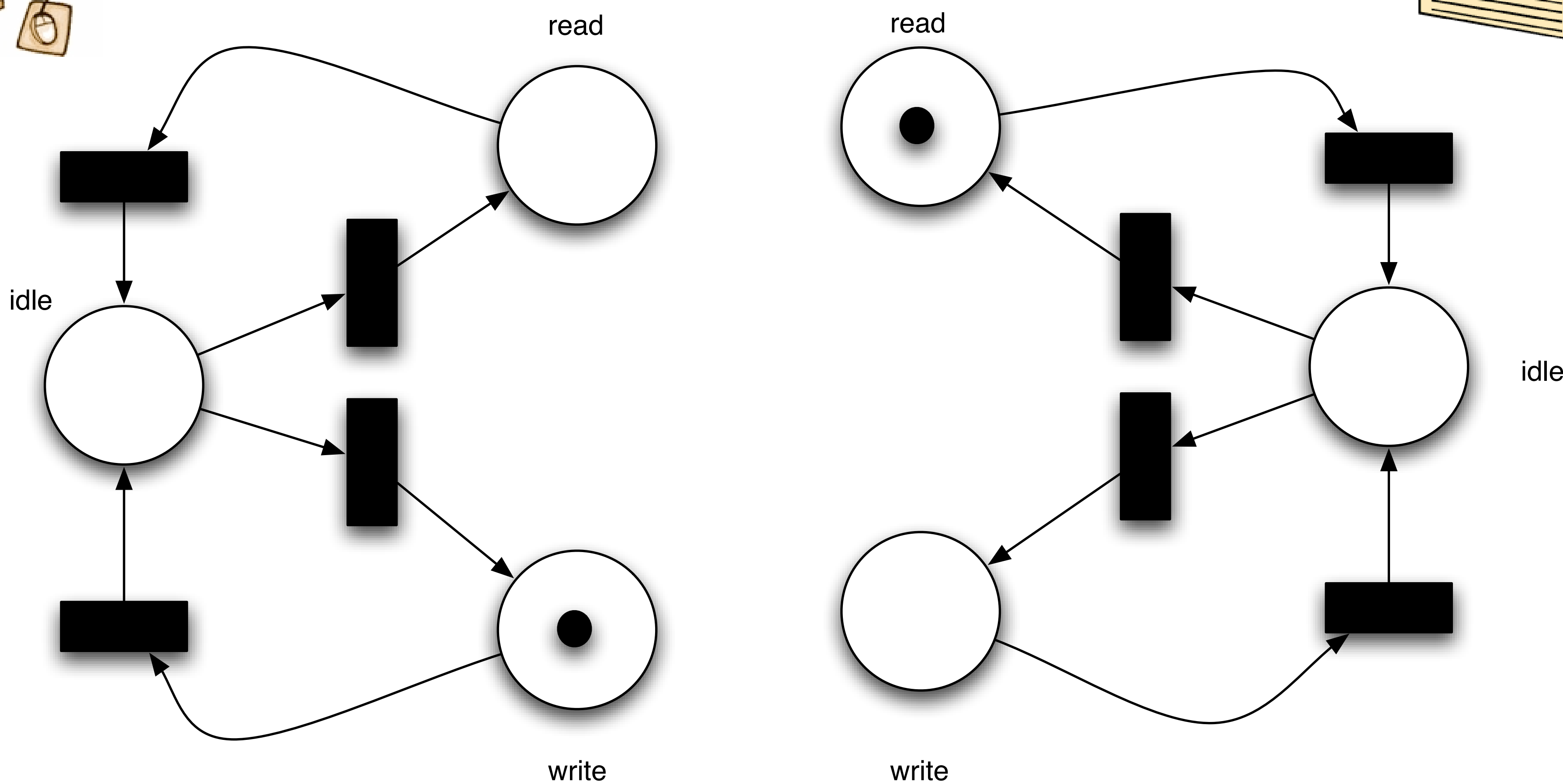


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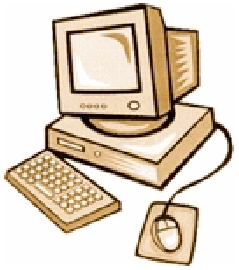
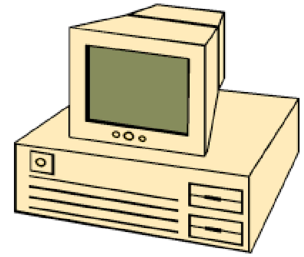
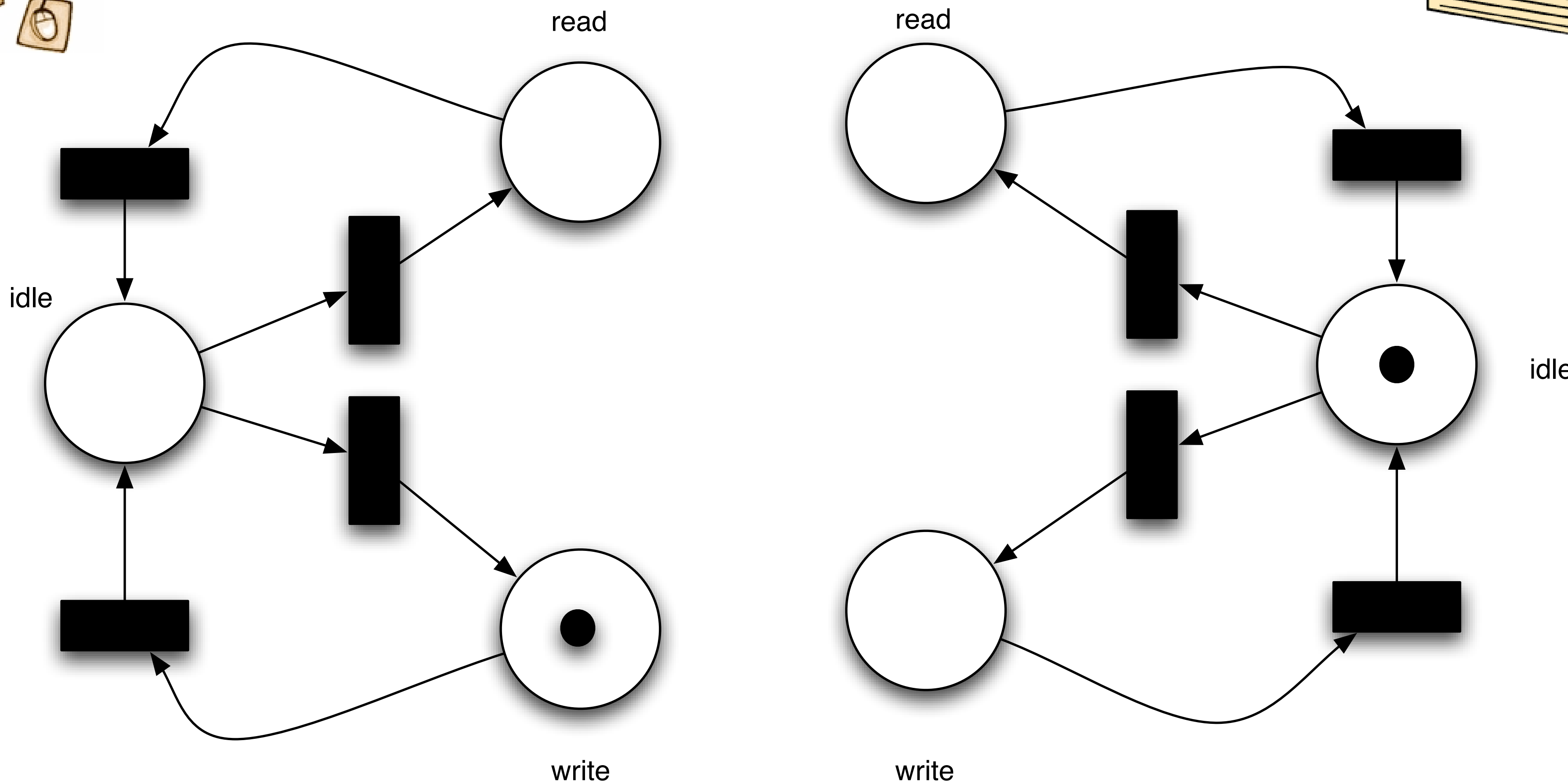
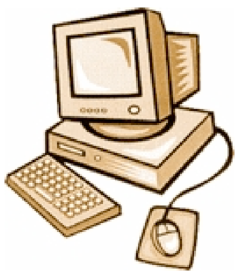


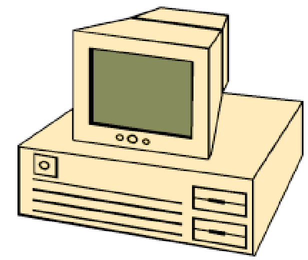


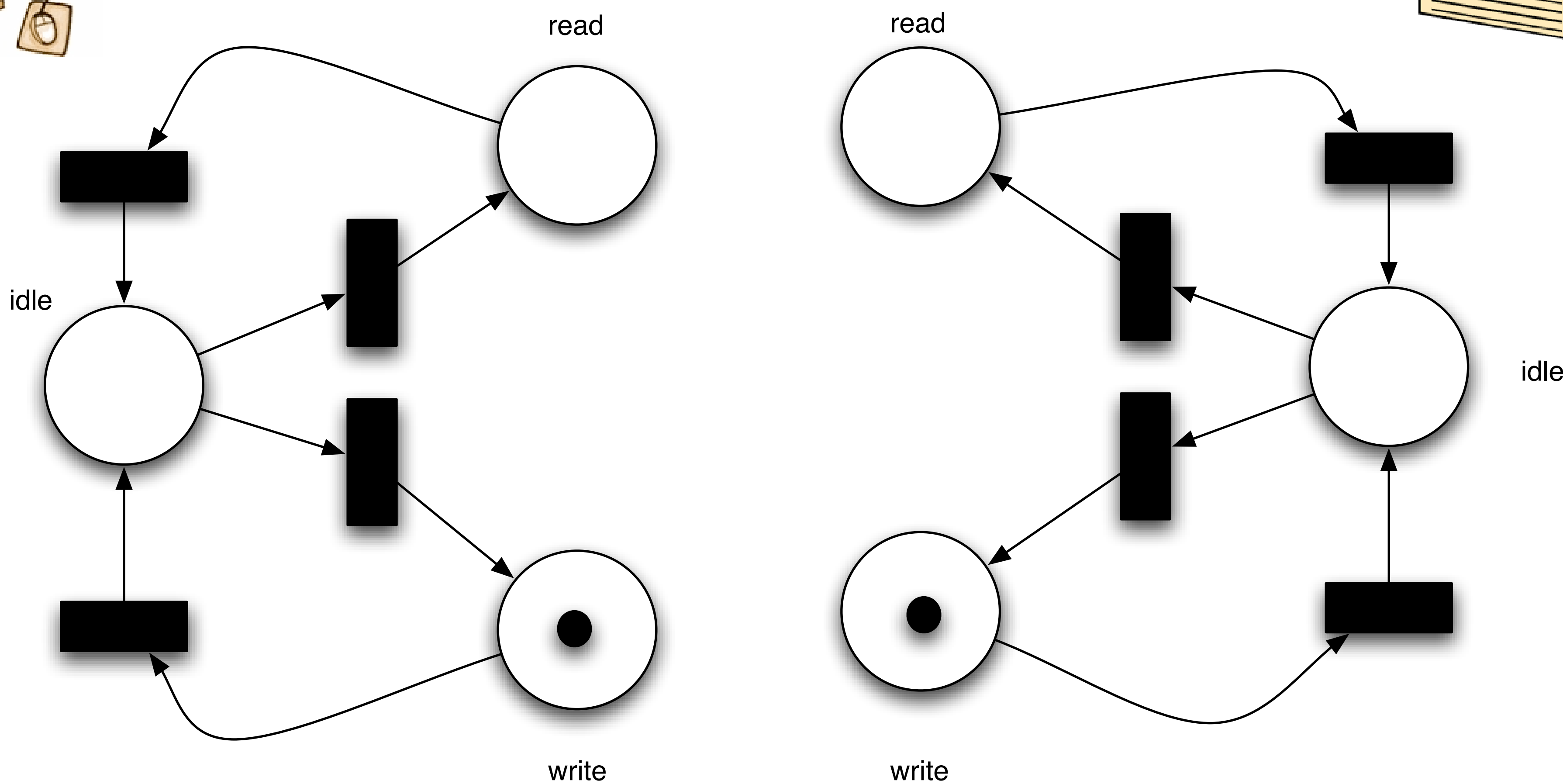



The token tells us the state of the process

# Example 1

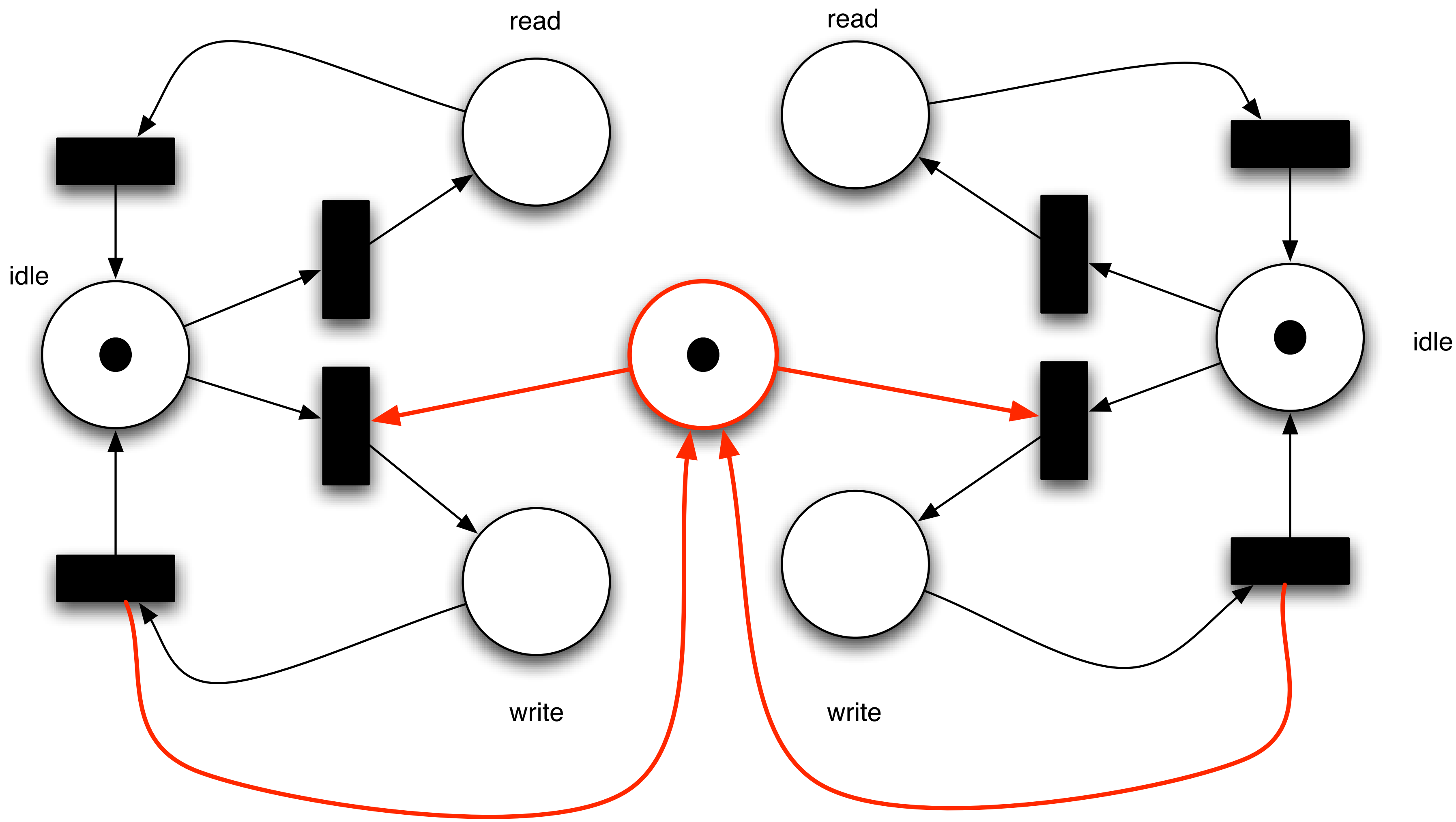






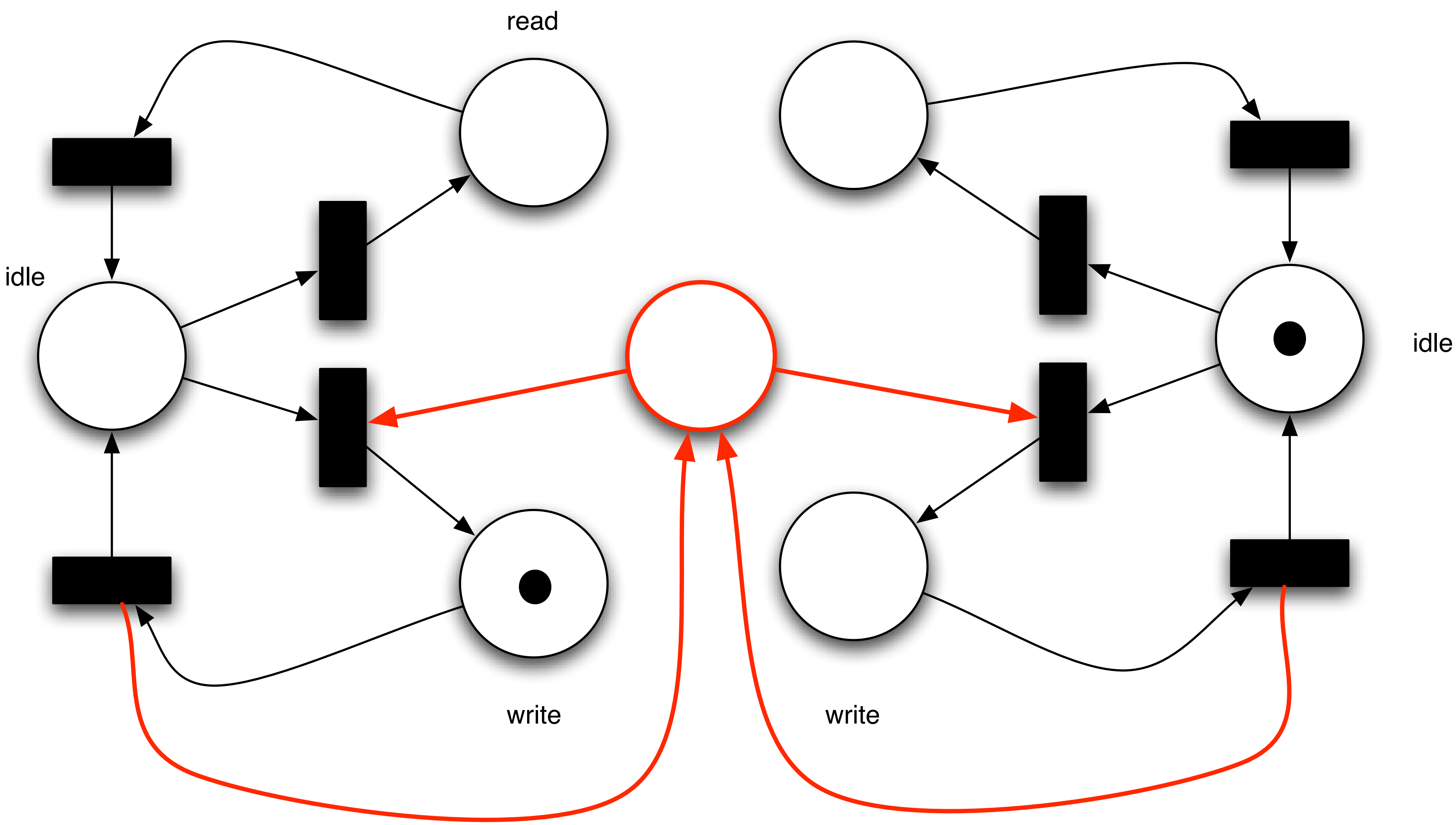

The token tells us the state of the process

# Example 1




Add a lock to ensure mutual exclusion

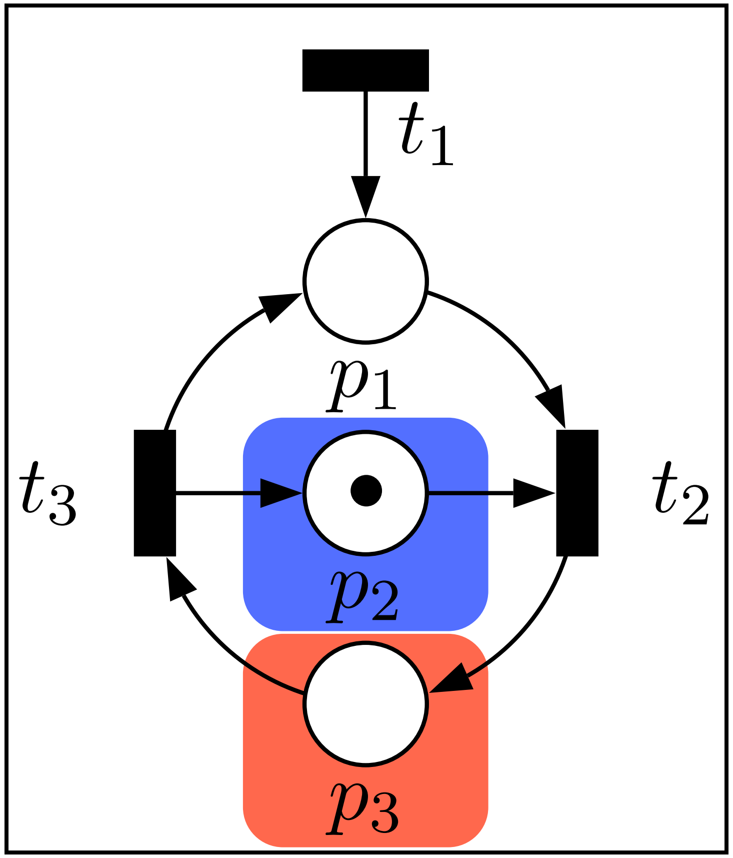
# Example 1



# Example 2

```
mutex M ;

Process P {
    repeat {
        take M ;
        critical ;
        release M ;
    }
}
```



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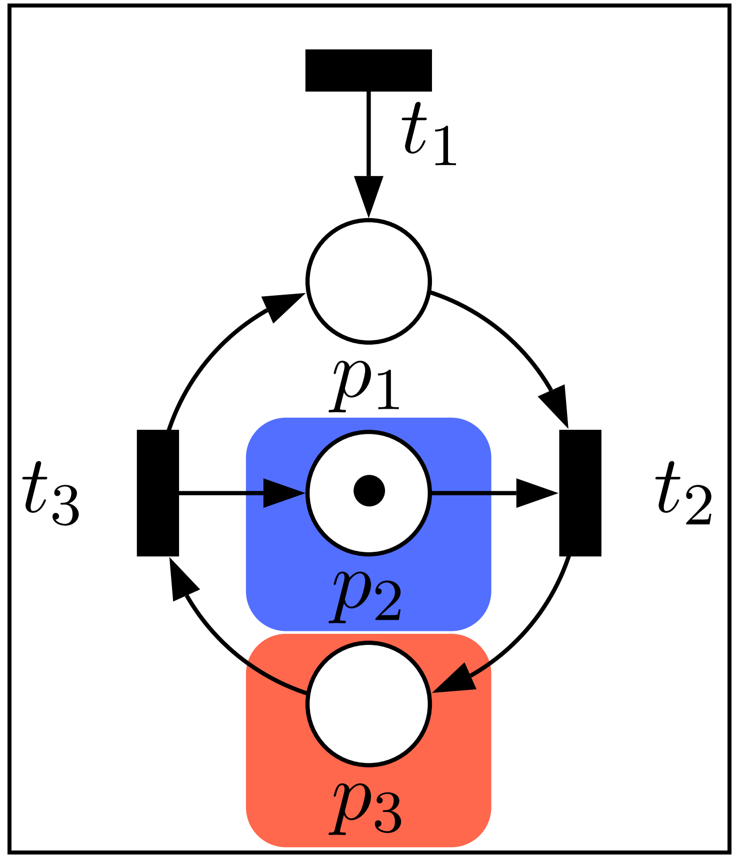



Here, we have applied a counting abstraction

# Plan of the talk

- Preliminaries
- Tools for the analysis of PN
  - reachability tree and reachability graph
  - place invariants
  - Karp & Miller and the coverability set
- The coverability problem
- More on PN: extensions...
- Conclusion

# Plan of the talk

Detailed coverage

- Preliminaries
- Tools for the analysis of PN
  - reachability tree and reachability graph
  - place invariants
  - Karp & Miller and the coverability set
- The coverability problem

Survey

- More on PN: extensions...
- Conclusion

# Preliminaries

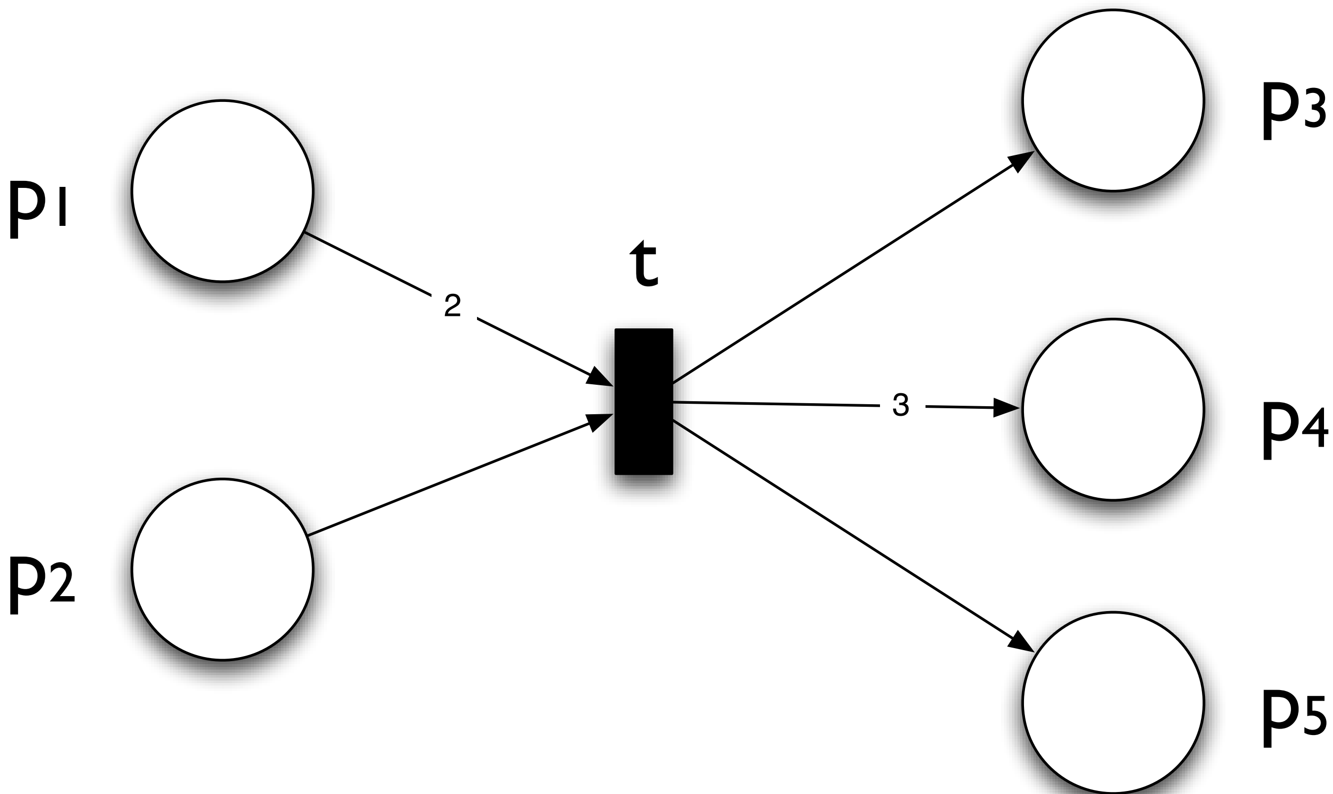
# Formal definition

- A Petri net is a tuple $\langle P, T \rangle$ where:
  - P is the (finite) set of places
  - T is the (finite) set of transitions. Each transition t is a tuple $\langle I, O \rangle$ where:
    - I: is a function s.t. t consumes I(p) tokens in each place p
    - O is a function s.t. t produces O(p) tokens in each place p

# Example

$I(p_1)=2$ $I(p_2)=1$ $I(p_3)=0$ $I(p_4)=0$ $I(p_5)=0$

$O(p_1)=0$ $O(p_2)=0$ $O(p_3)=1$ $O(p_4)=3$ $O(p_5)=1$



# Markings

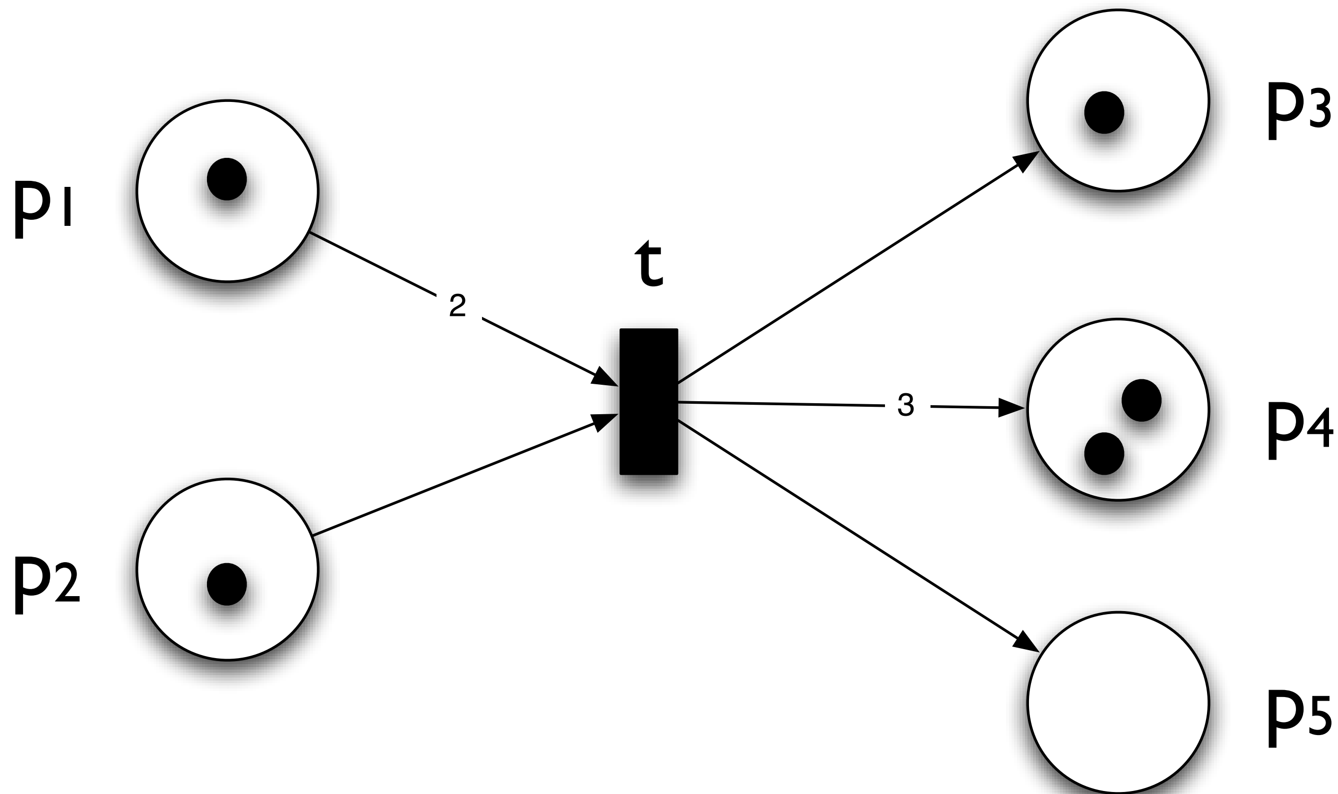
- The distribution of the tokens in the places is formalised by the notion of marking, which can be seen:
  - either as a function $m$, s.t. $m(p)$ is the number of tokens in place $p$
  - or as a vector $m = \langle m_1, m_2, \ldots m_n \rangle$ where $m_i$ is the number of tokens in place $p_i$

# Example

$$m = \langle 1,1,1,2,0 \rangle$$

$$m = \langle p_1, p_2, p_3, 2p_4 \rangle$$

$$m(p_1)=1, m(p_2)=1, m(p_3)=1, m(p_4)=2, m(p_5)=0$$



# Firing a transition

- A transition t = $\langle I, O \rangle$ can be fired from m iff for any place p:

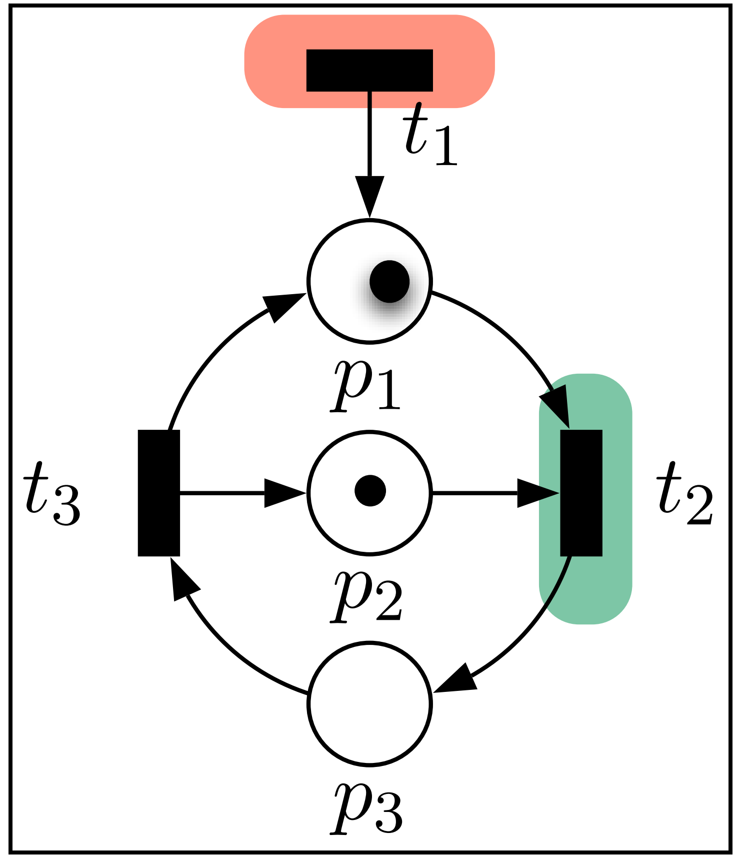
$$m(p) \geq I(p)$$

- The firing transforms the marking m into a marking m' s.t. for any place p:

$$m'(p) = m(p) - I(p) + O(p)$$

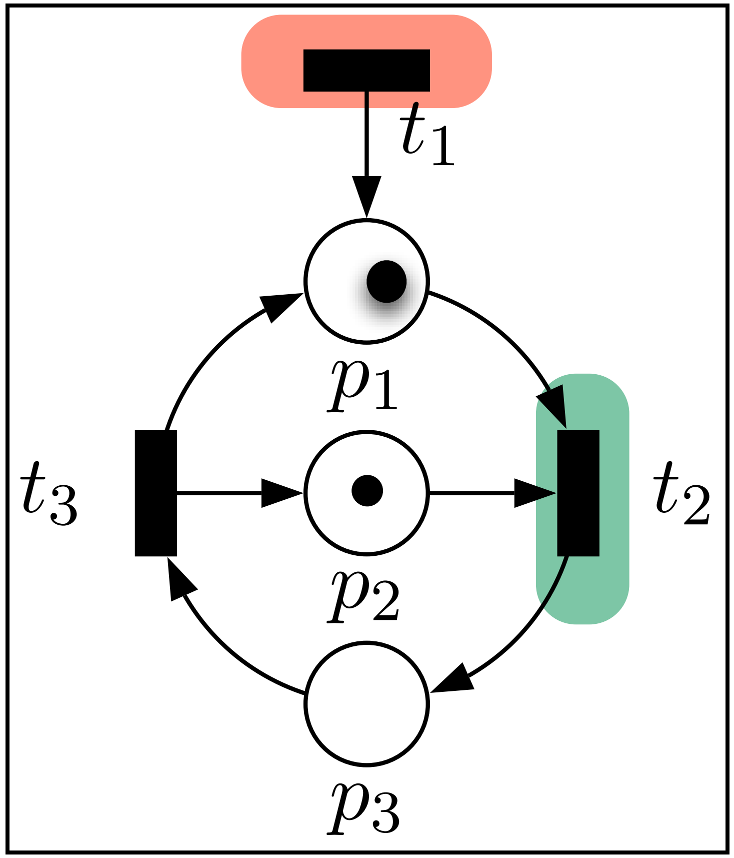
- Notation: $m \rightarrow m'$
- Notation: $Post(m) = \{m' \mid m \rightarrow m'\}$

# Example



# Example

Post( $\langle 1, 1, 0 \rangle$ )=
{ $\langle 2, 1, 0 \rangle$ , $\langle 0, 0, 1 \rangle$ }



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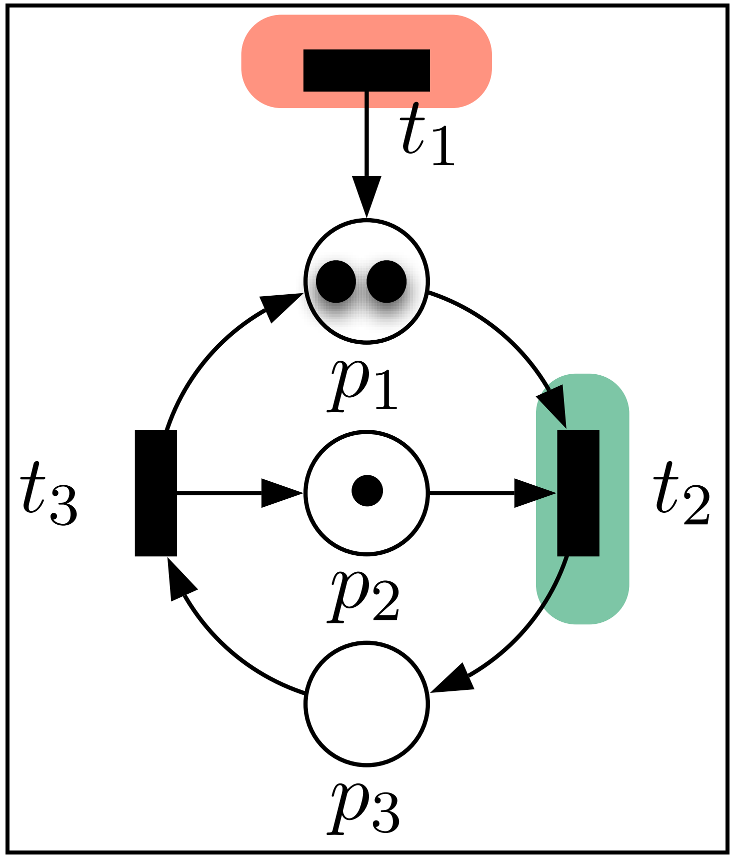

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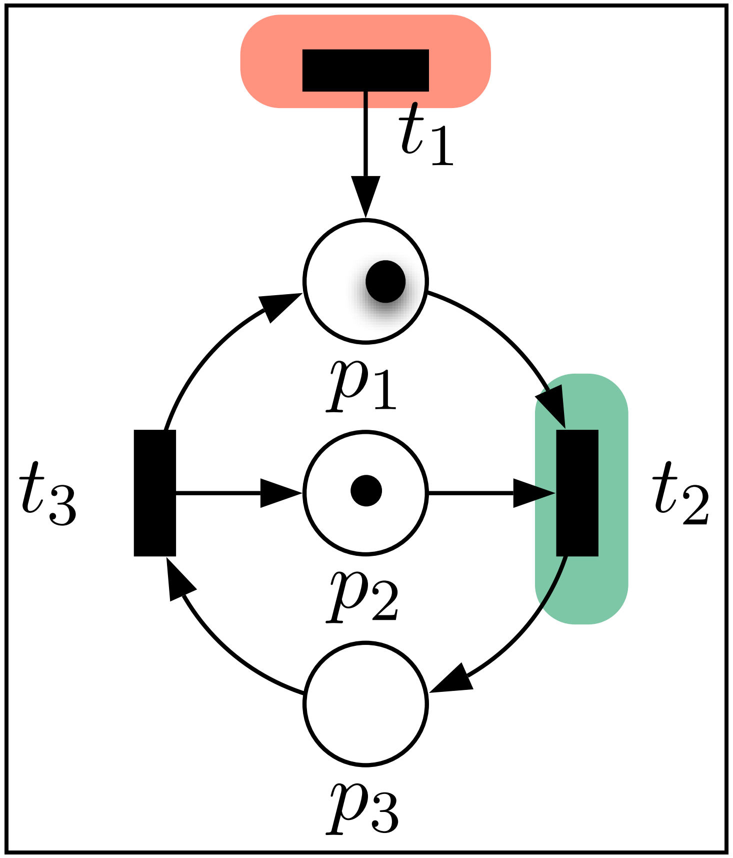

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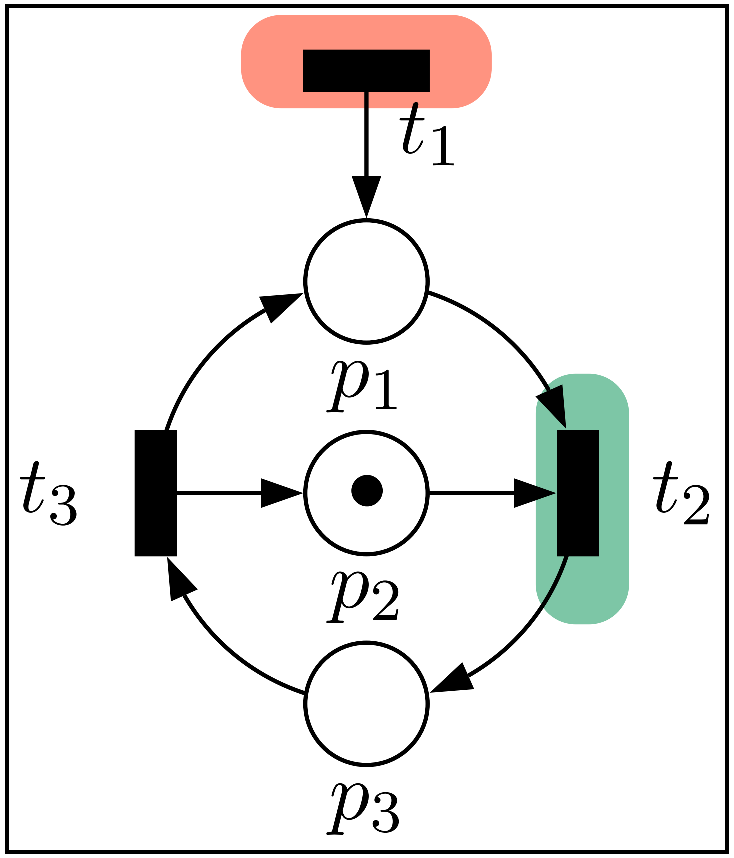
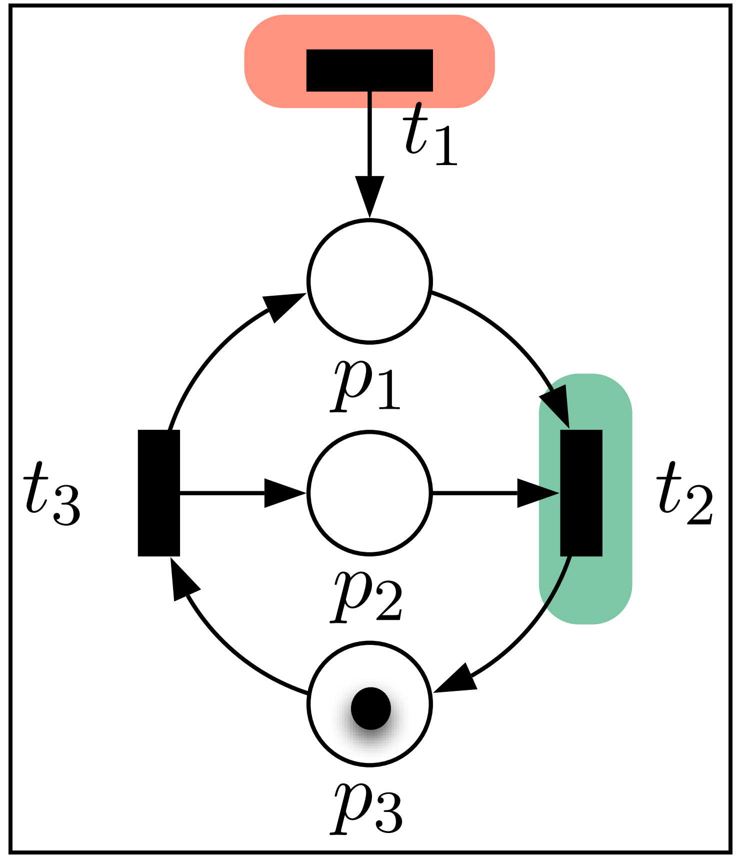


# Example

$$\text{Post}(\ \langle 1,1,0\rangle\ ) = \{\ \langle 2,1,0\rangle\ ,\ \langle 0,0,1\rangle\ \}$$



# Initial marking
# Reachable markings

- All PN are equipped with an initial marking $m_0$
- If two markings m and m' are s.t.:

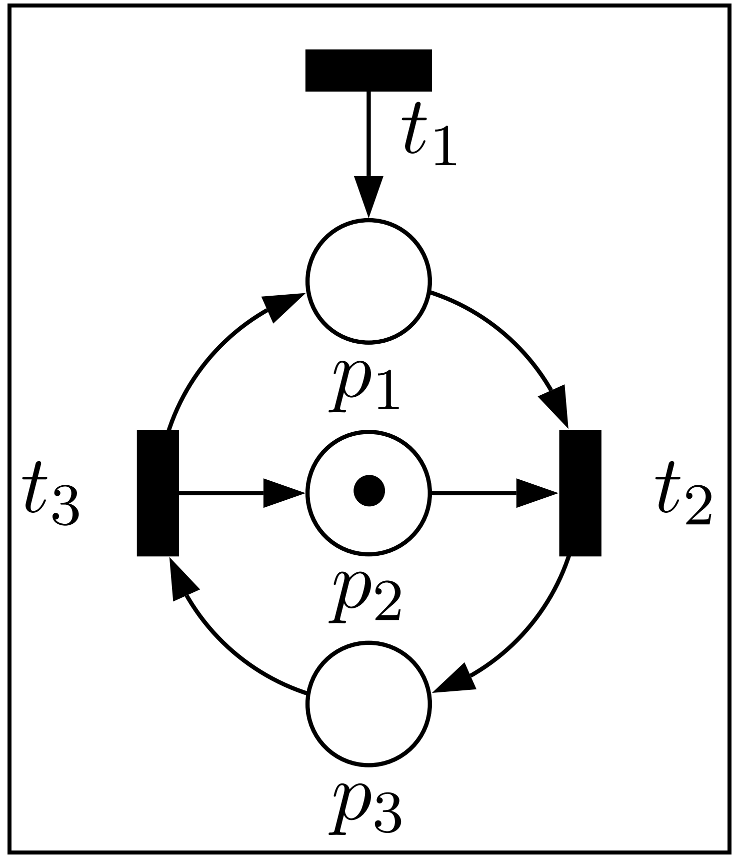
  $$m \rightarrow m_1 \rightarrow m_2 \rightarrow \cdots \rightarrow m'$$

  Then m' is reachable from m
- Let N be a PN with initial marking $m_0$:

  $$\text{Reach}(N) = \{m \text{ reachable from } m_0\}$$

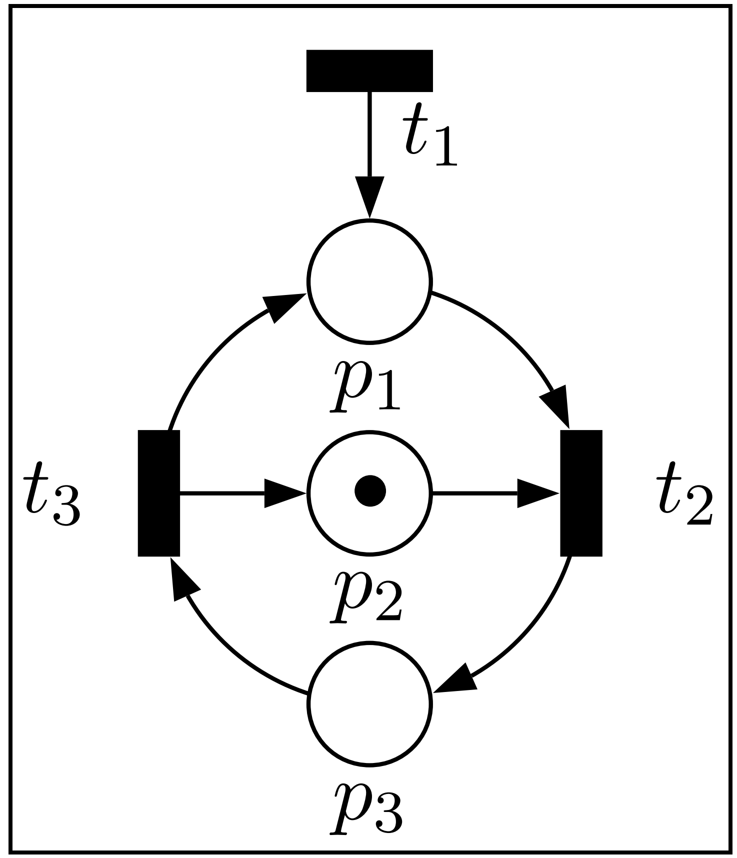
  is the set of reachable markings of N.

# Example



# Example

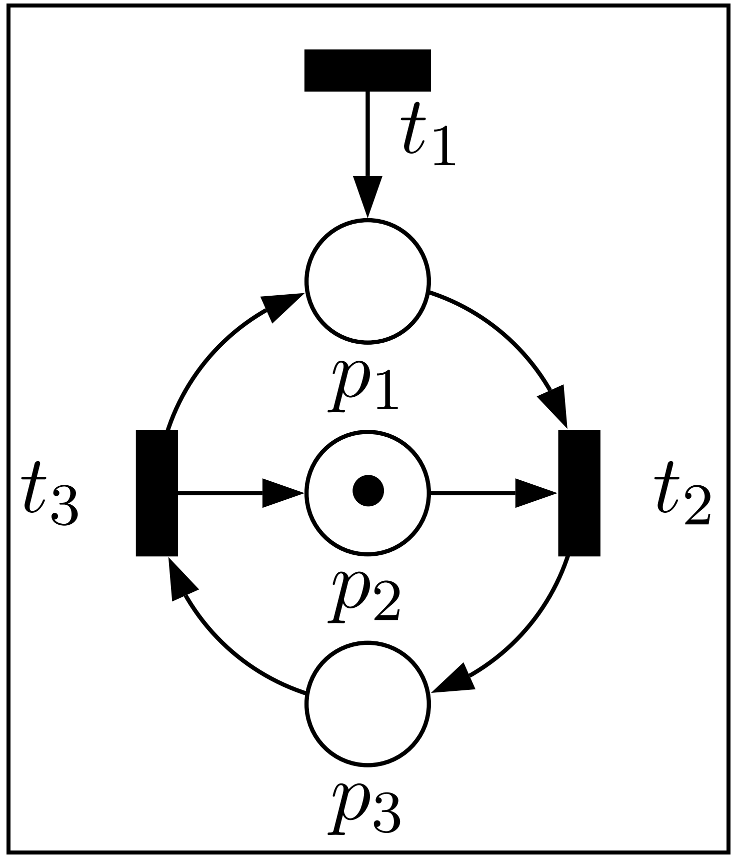
$$\mathrm{Reach}(\mathcal{N}) = \{\langle i,1,0\rangle \mid i \in \mathbb{N}\} \cup \{\langle i,0,1\rangle \mid i \in \mathbb{N}\}$$



# Example

$$\mathrm{Reach}(\mathcal{N}) = \{\langle i, 1, 0\rangle \mid i \in \mathbb{N}\} \cup \{\langle i, 0, 1\rangle \mid i \in \mathbb{N}\}$$

This set allows us to prove that the mutual exclusion is indeed enforced



# Ordering on markings

- Markings can be compared thanks to $\preccurlyeq$:

  $m \preccurlyeq m'$ iff for any place p: $m(p) \leqslant m'(p)$

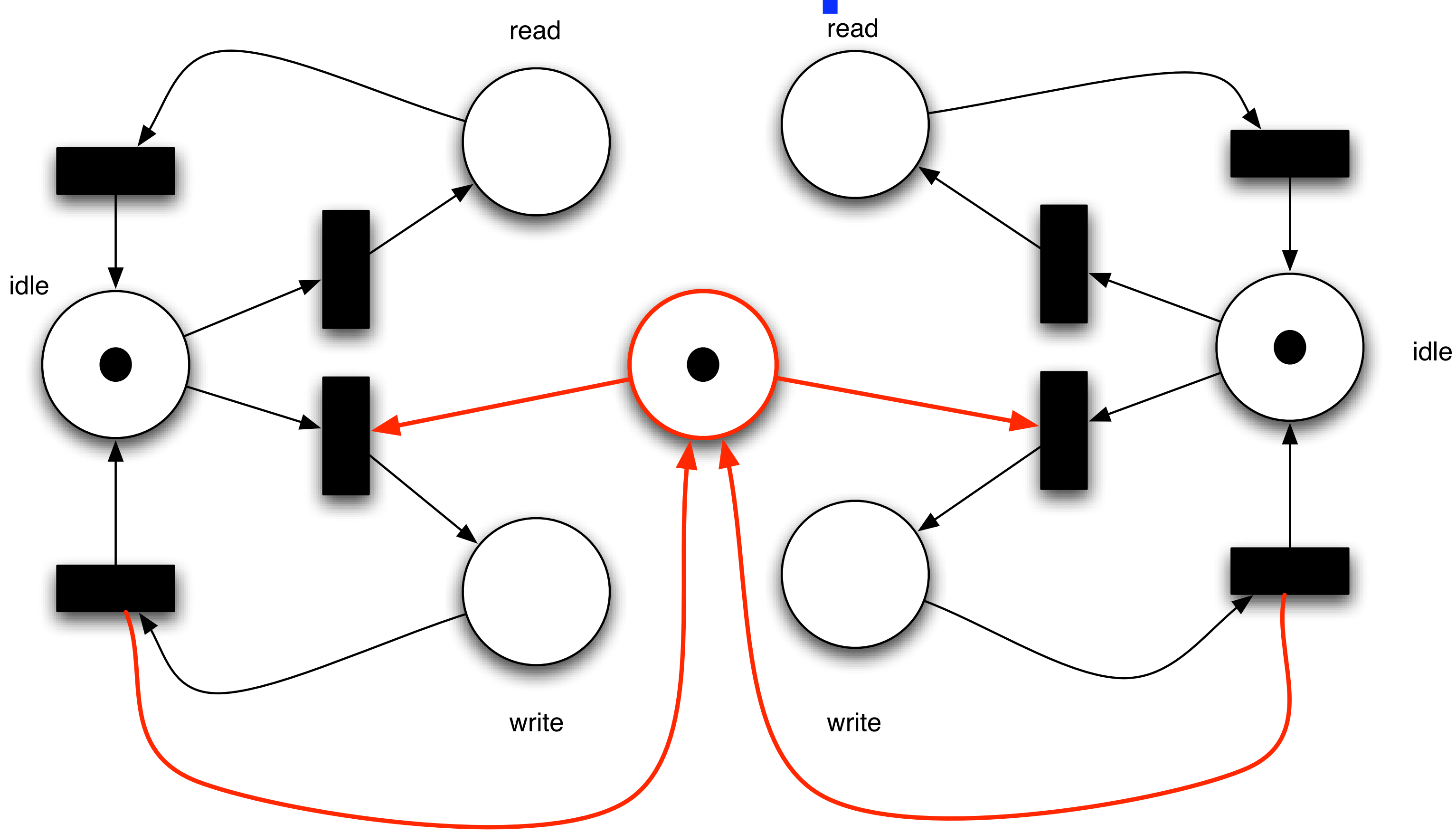
  $m \prec m'$ iff $m \preccurlyeq m'$ and $m \neq m'$

- Examples:
  - $\langle 1, 0, 0 \rangle \prec \langle 1, 1, 0 \rangle \preccurlyeq \langle 1, 1, 0 \rangle \preccurlyeq \langle 5, 7, 2 \rangle$
  - $\langle 1, 0, 0 \rangle$ is not comparable to $\langle 0, 1, 0 \rangle$

# Questions on PN

- Meaningful questions about PN include:
  - Boundedness: is the number of reachable markings bounded ?
  - Place boundedness: is there a bound on the maximal number of tokens that can be created in a given place ?
  - Semi-liveness: is there a reachable marking from which a given transition can fire ?
  - Coverability

# Example



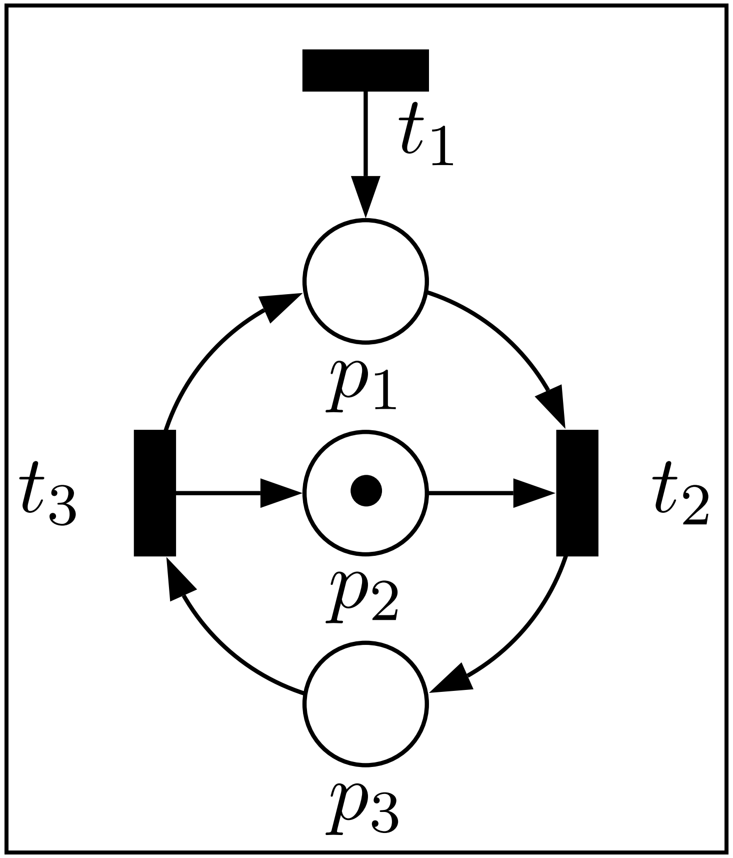

Bounded PN

All the places are bounded

All the transitions are semi-live

# Example

- Unbounded PN
- $p_2$ and $p_3$ are bounded
- $p_1$ is unbounded
- All the transitions are semi-live



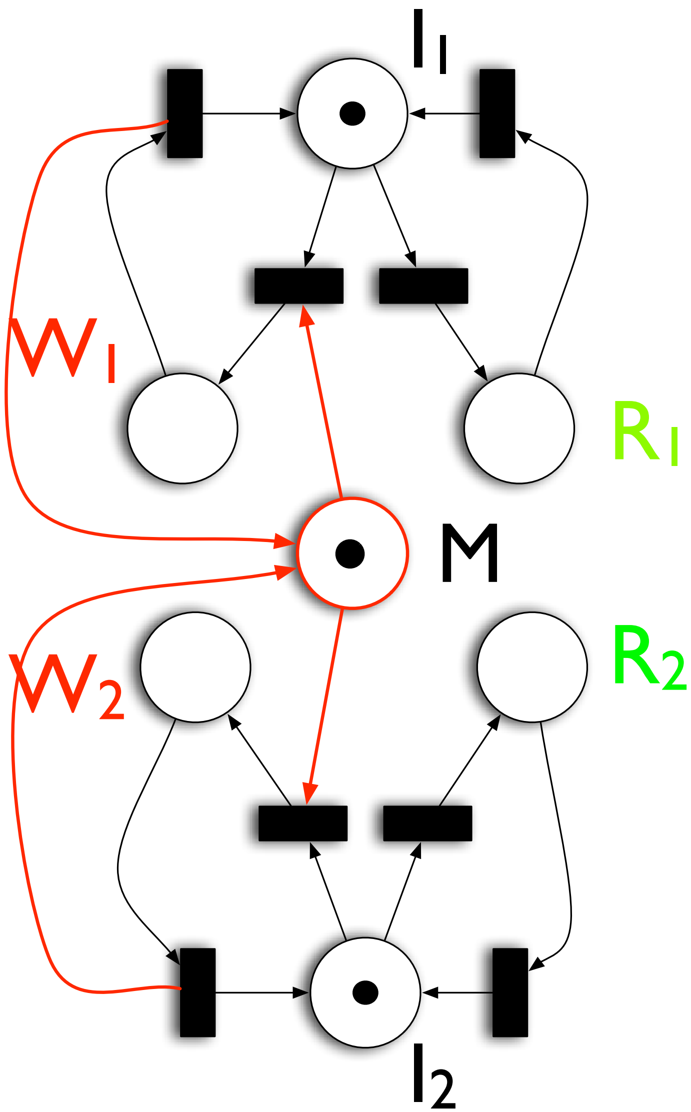
# Some tools for the analysis of PN

# Reachability tree and reachability graph

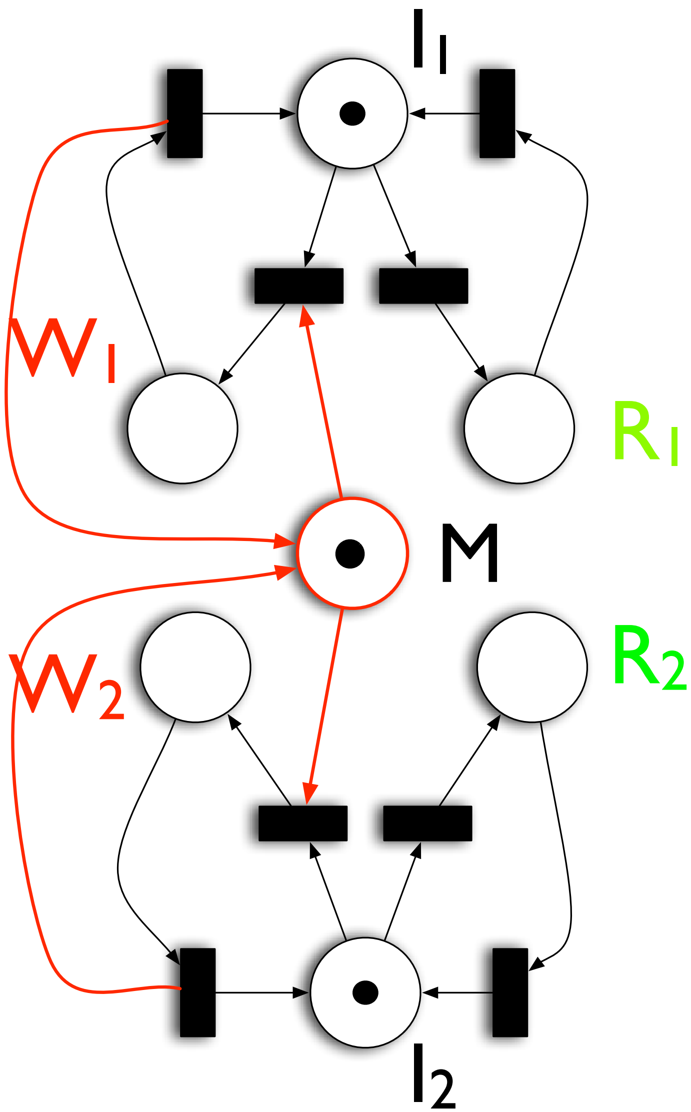
# Reachability Tree

- Idea:
  - the root is labeled by $m_0$
  - for each node labeled by m, create one child for each marking of Post(m)

# Reachability Tree

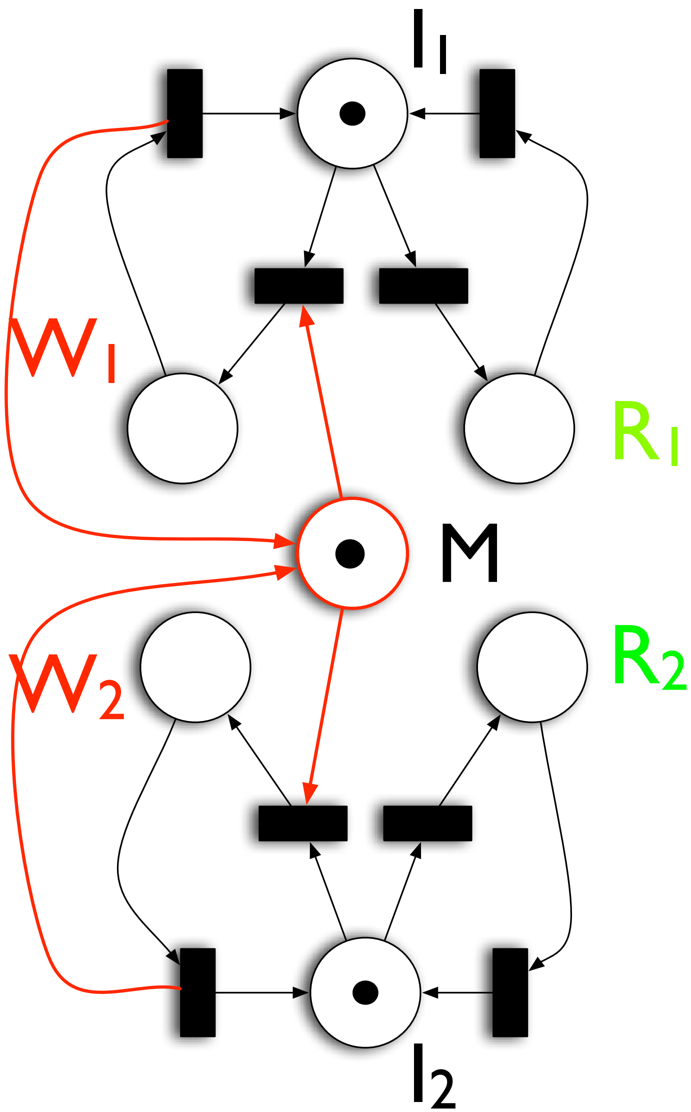


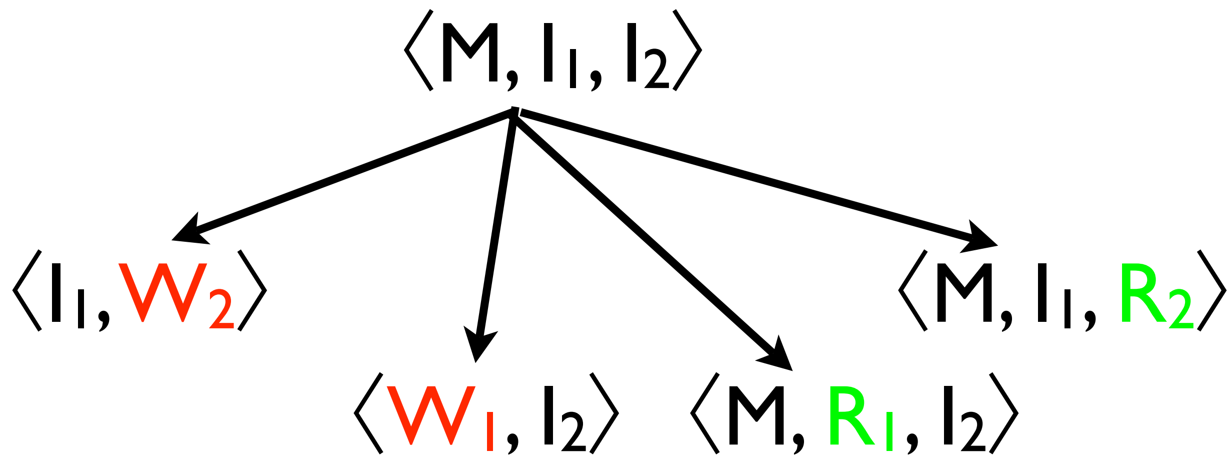
# Reachability Tree




$\langle M, I_1, I_2 \rangle$

# Reachability Tree



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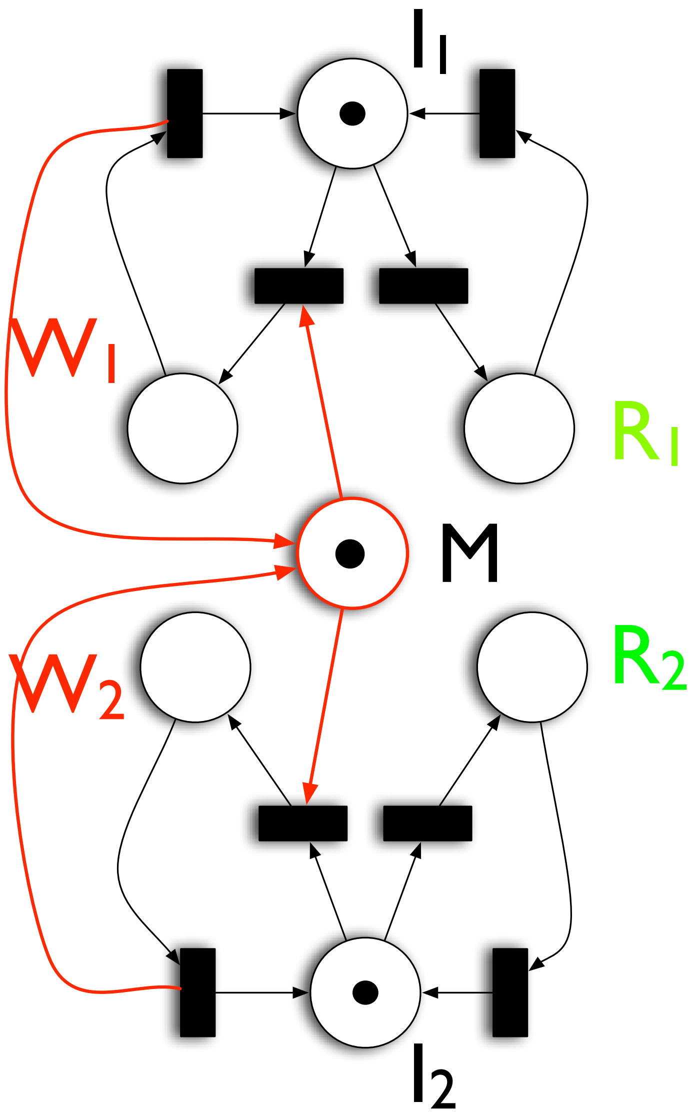


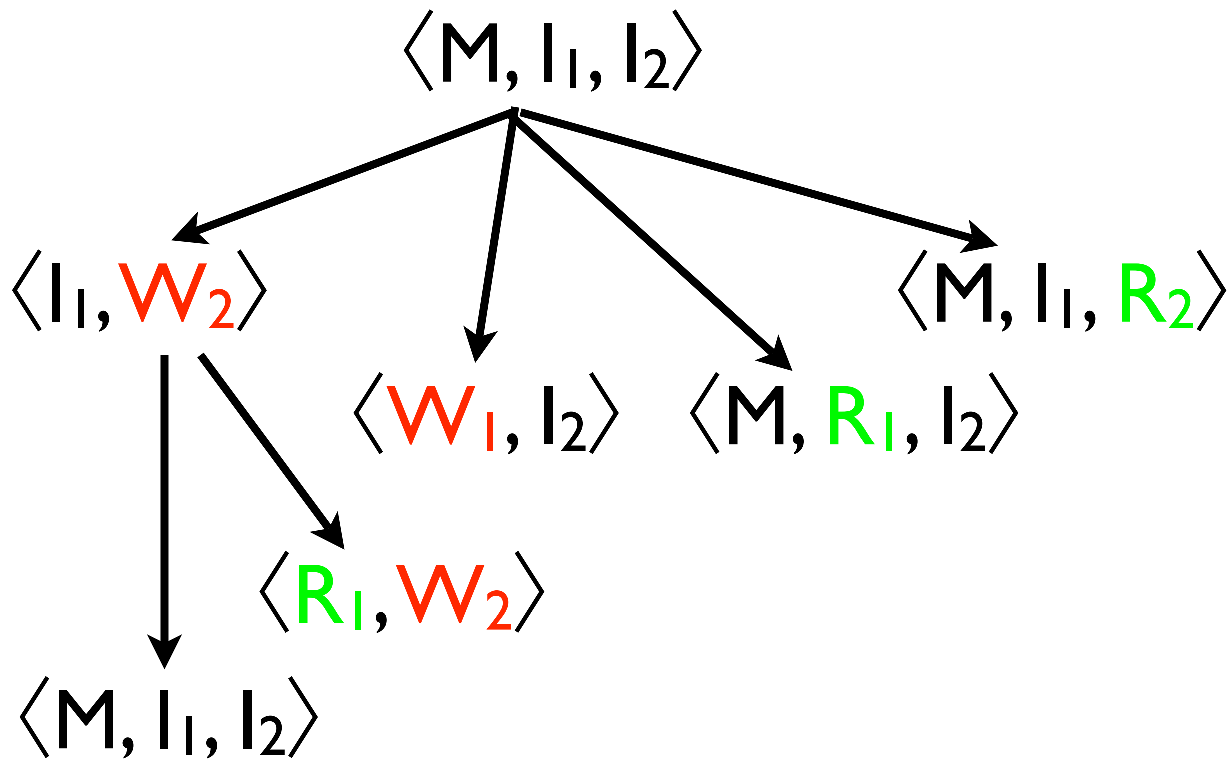

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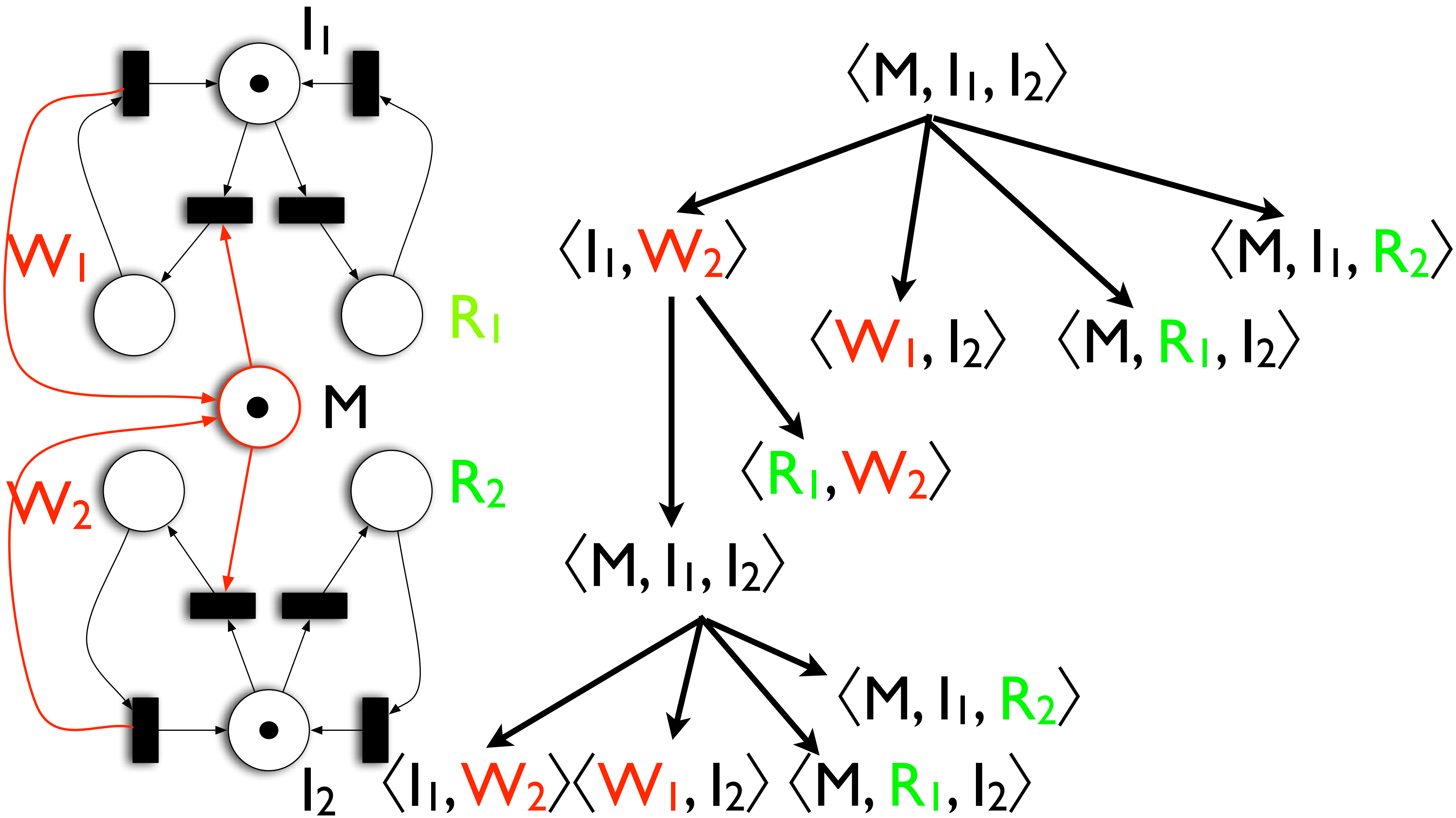

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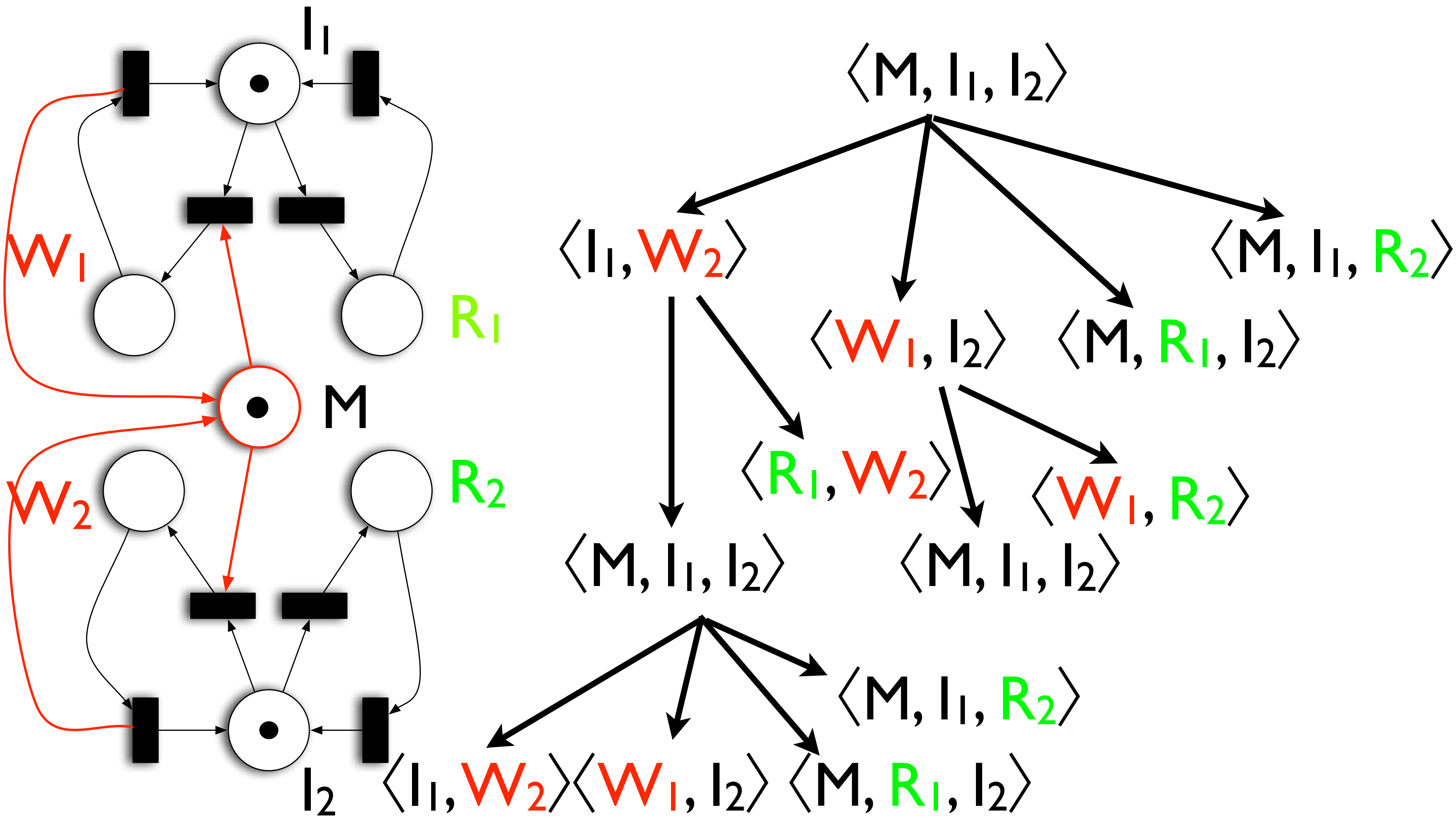

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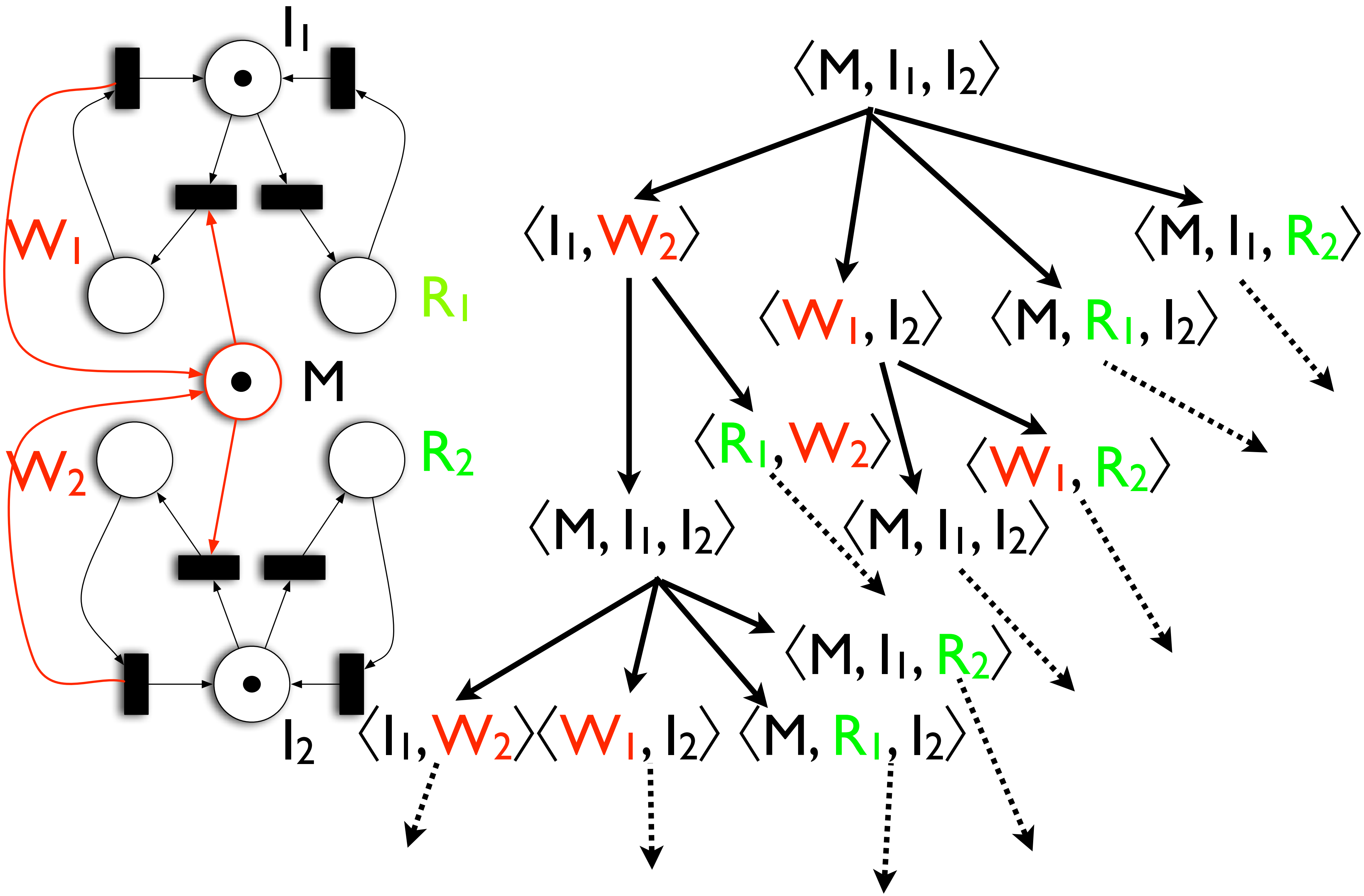
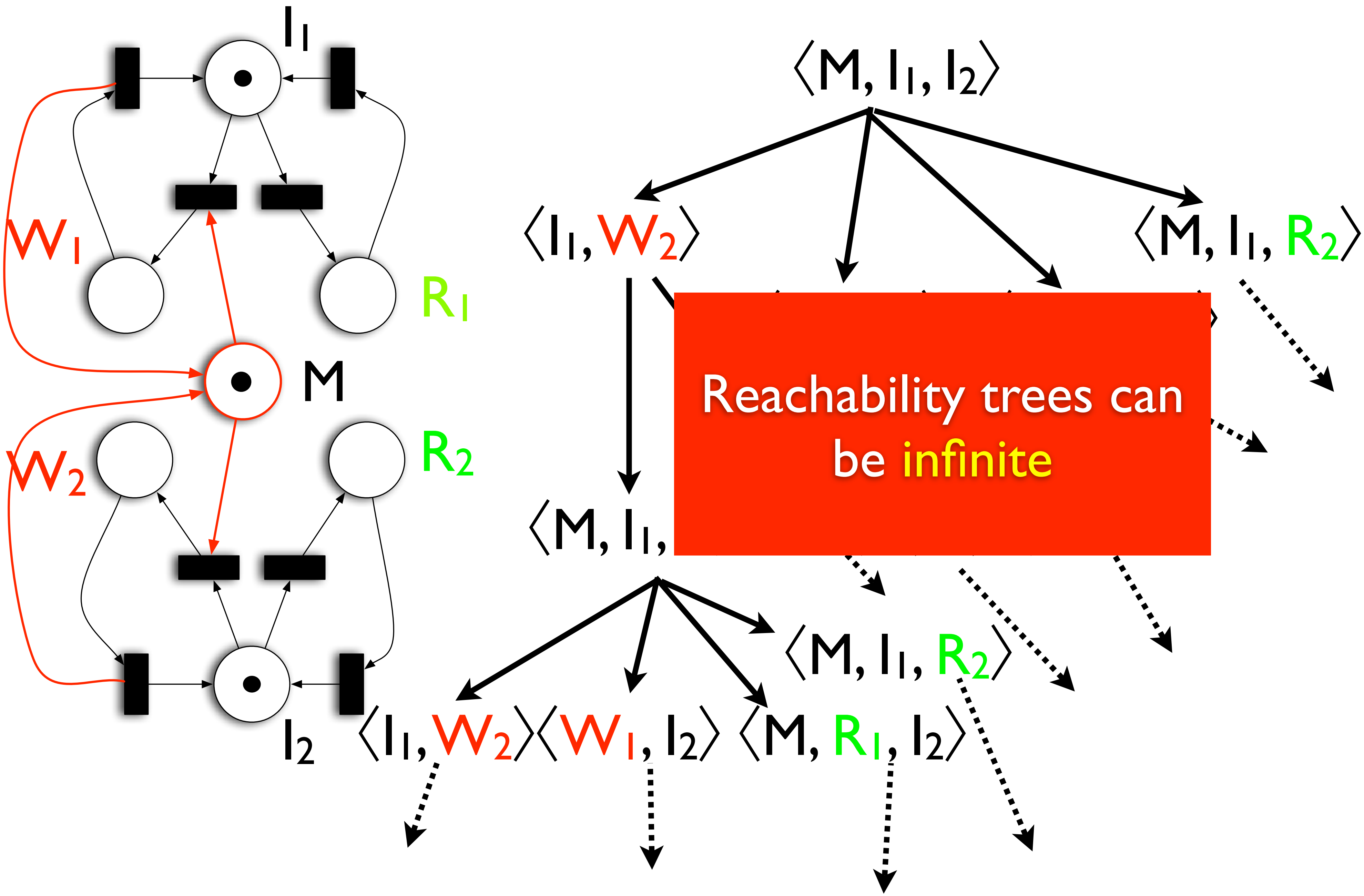


# Reachability Tree

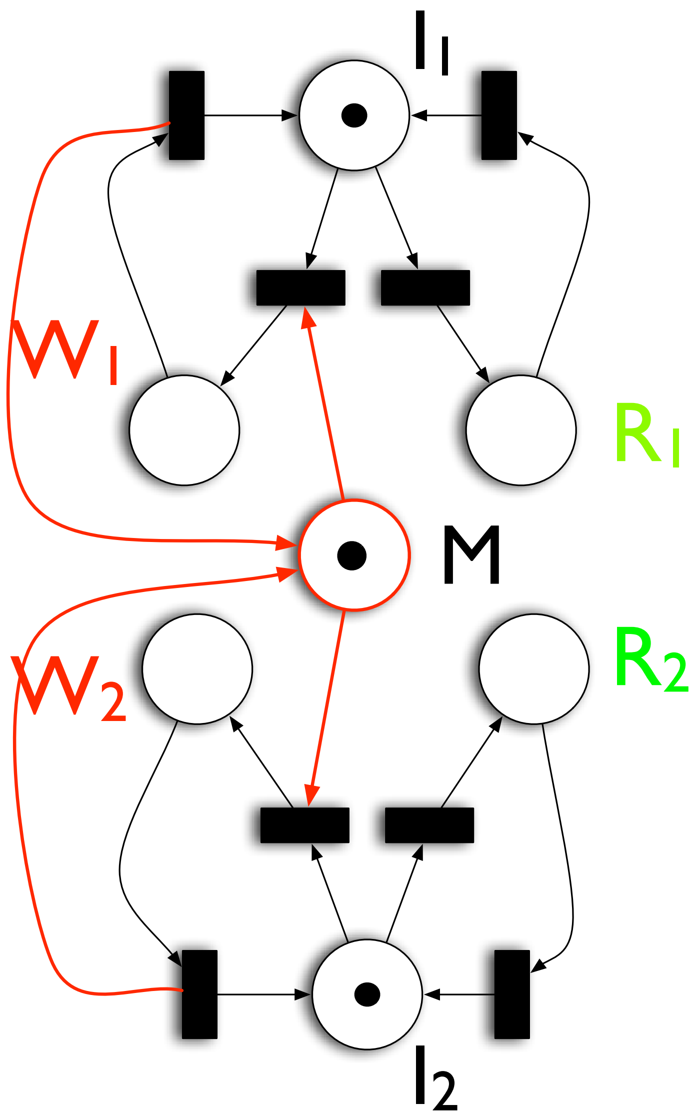



Reachability trees can be infinite

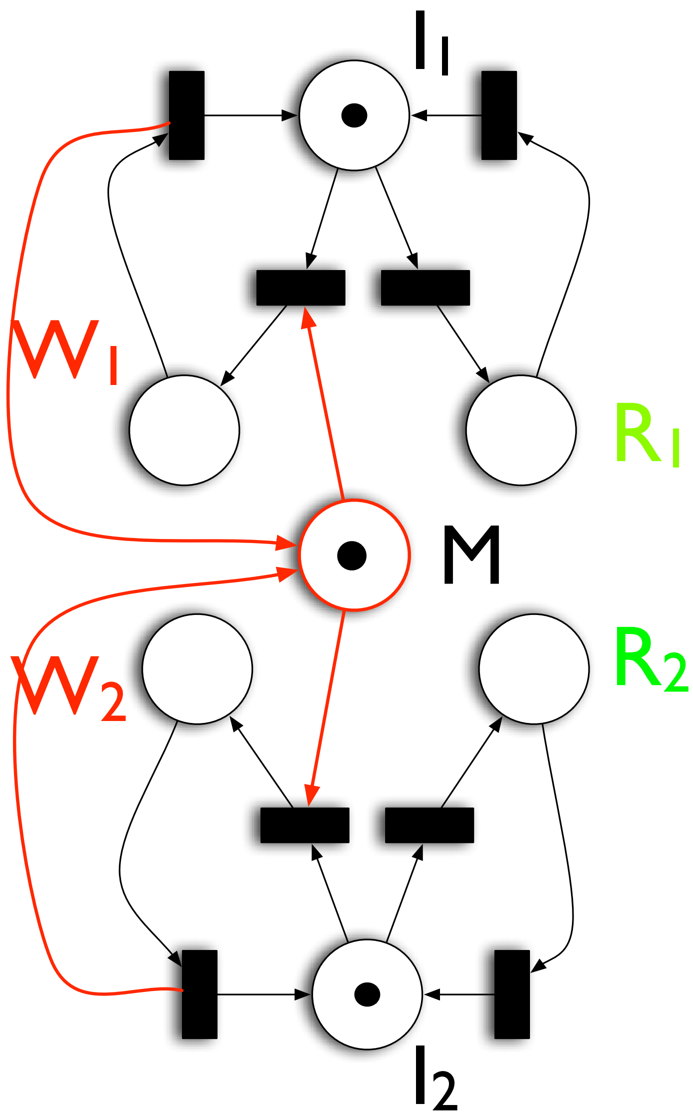
# Reachability graph

- Idea: build a node for each reachable marking and add an edge from m to m' if some transition transforms m into m'
  - remark: now, if we meet the same marking twice, we do not create a new node, but re-use the previously created node.

# Reachability graph

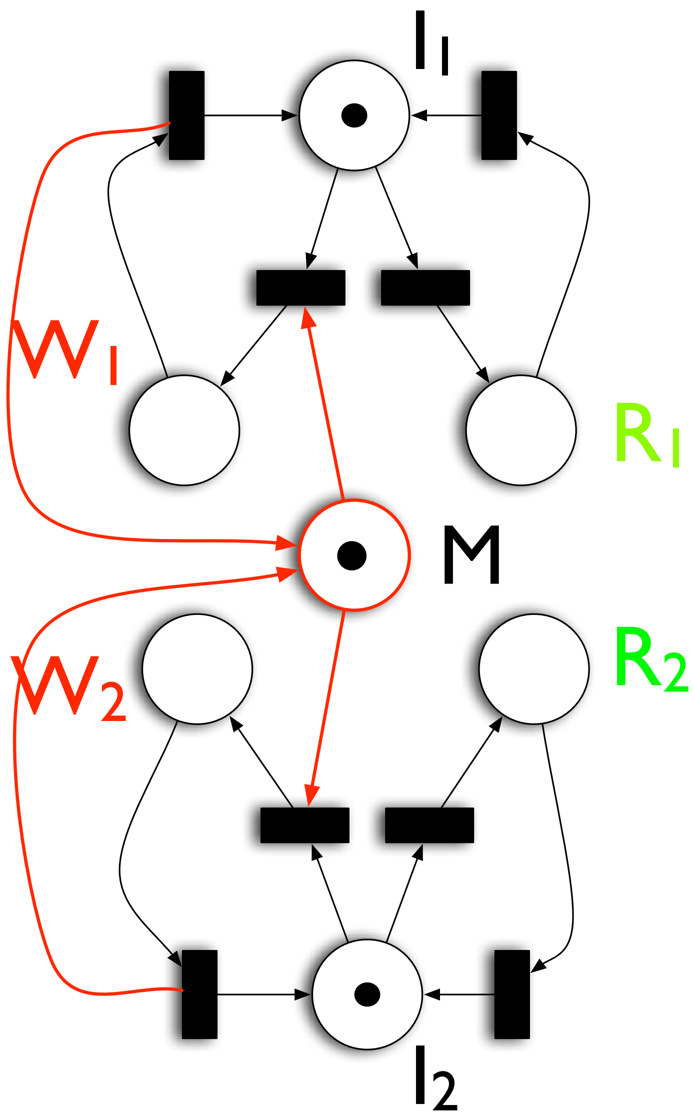


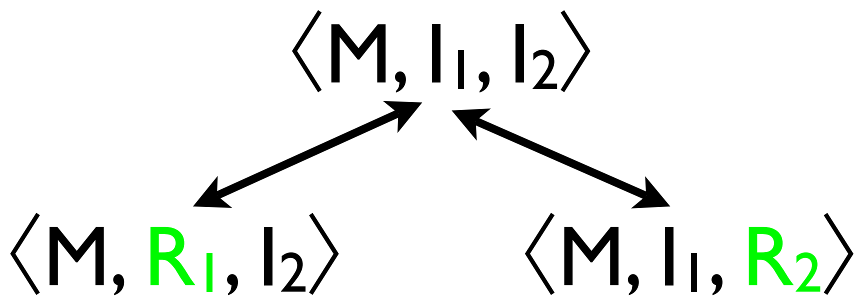
# Reachability graph




$\langle M, I_1, I_2 \rangle$

# Reachability graph



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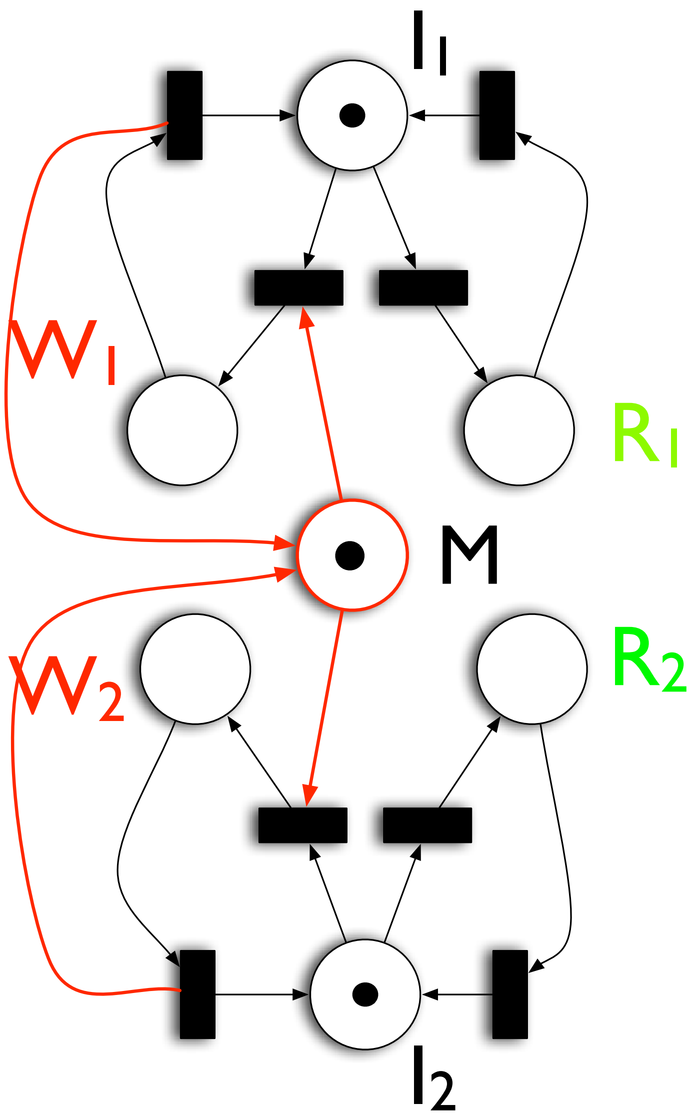


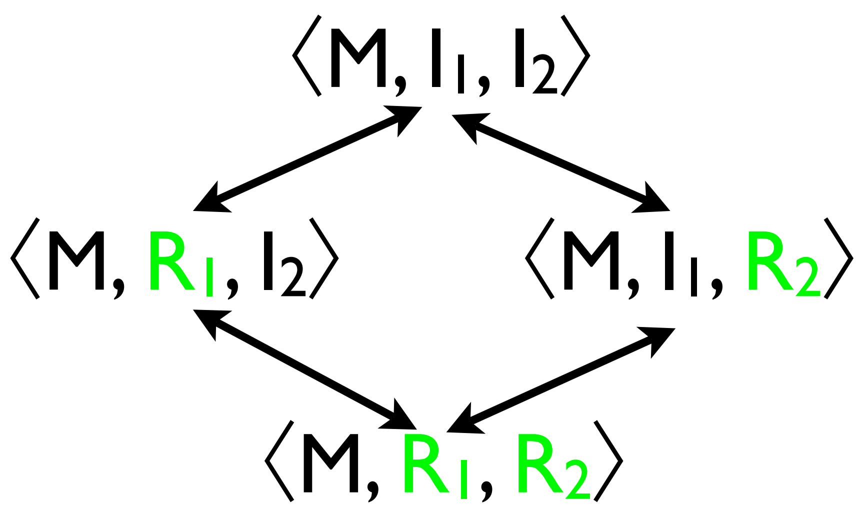

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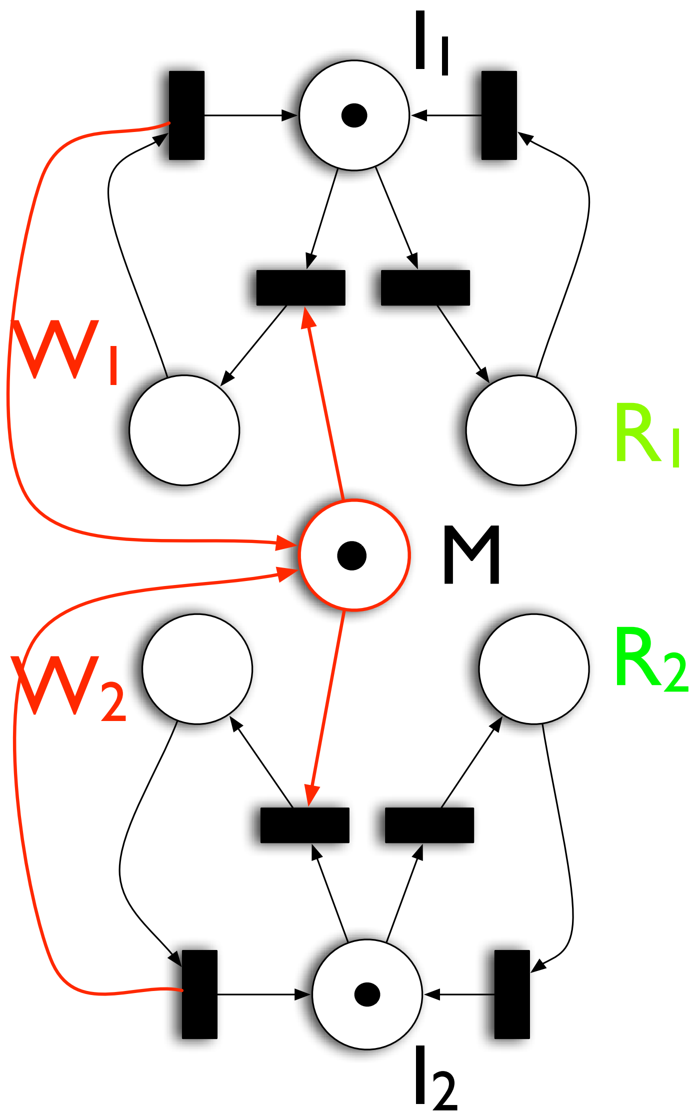
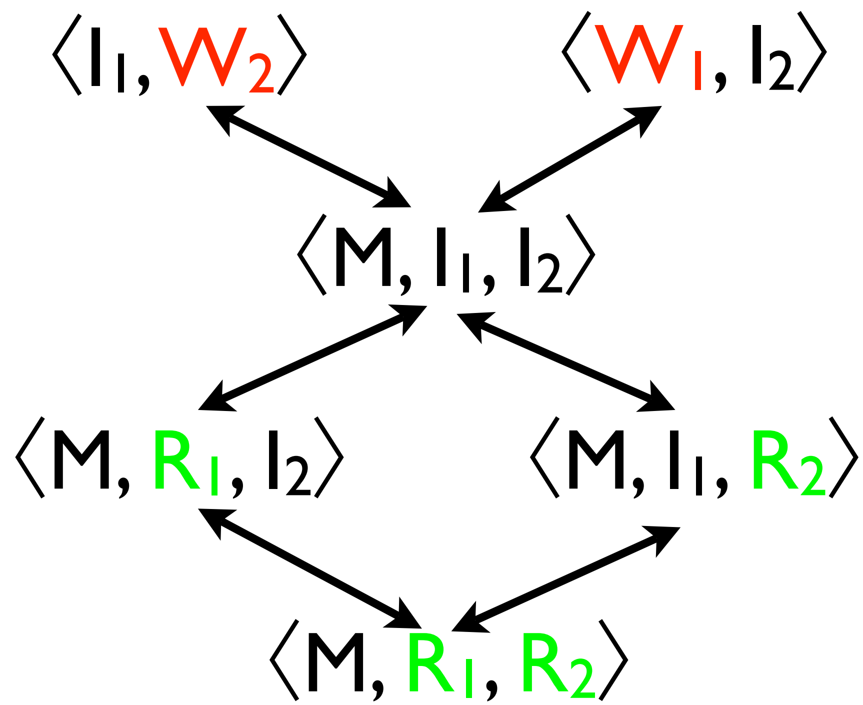
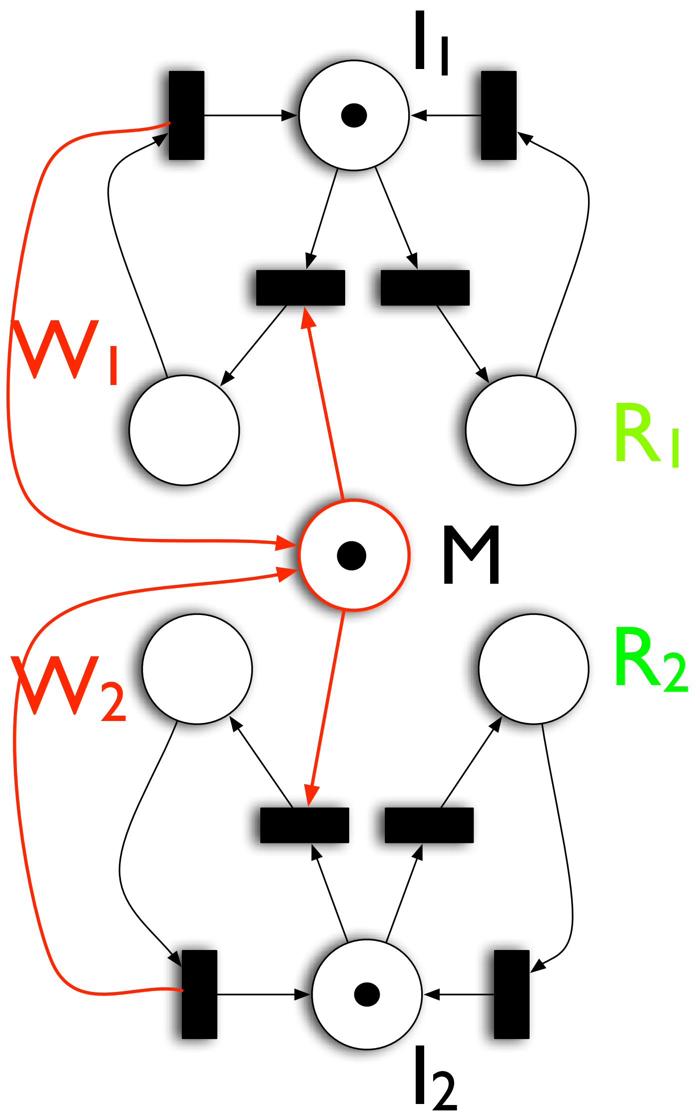
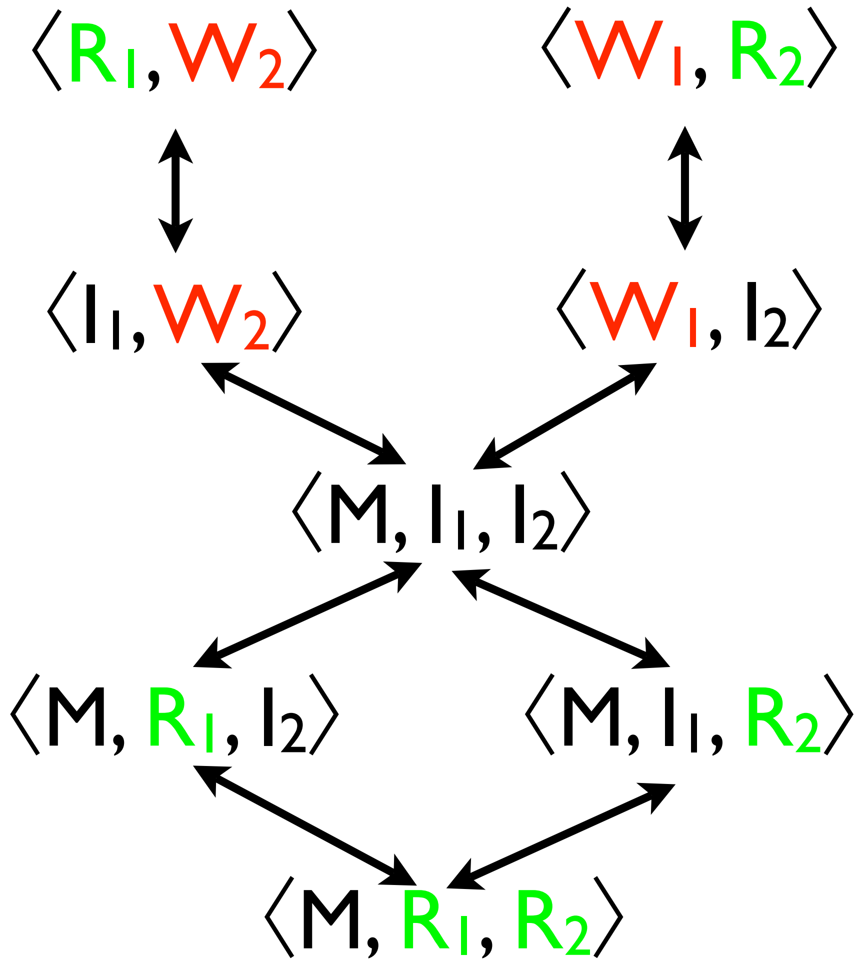


# Reachability graph



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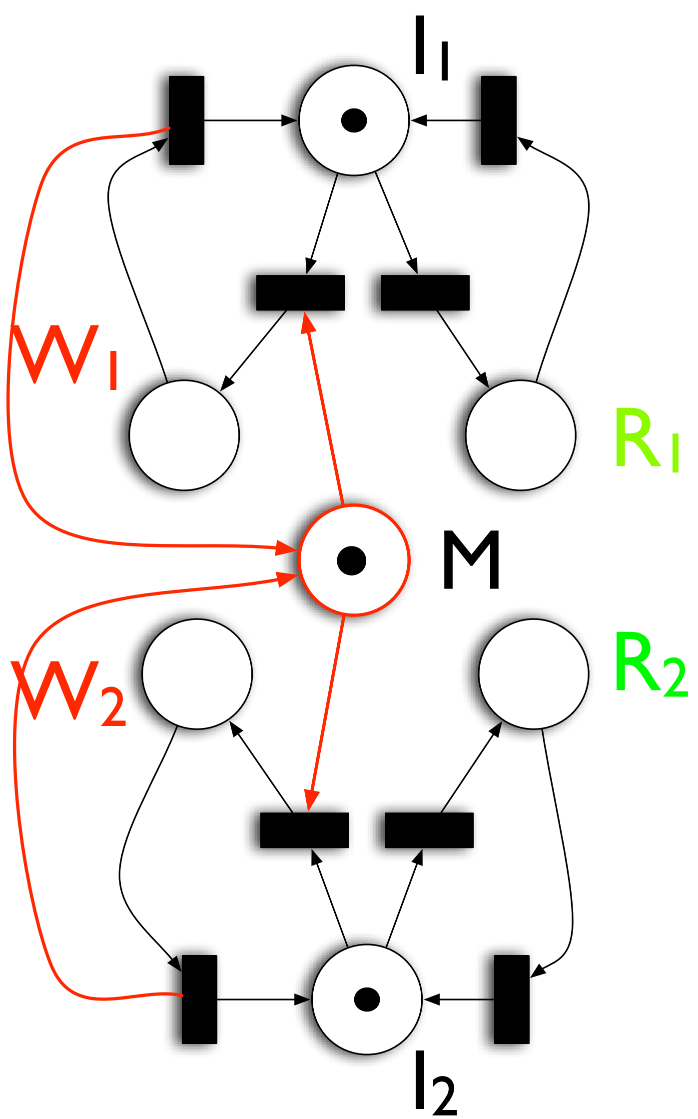
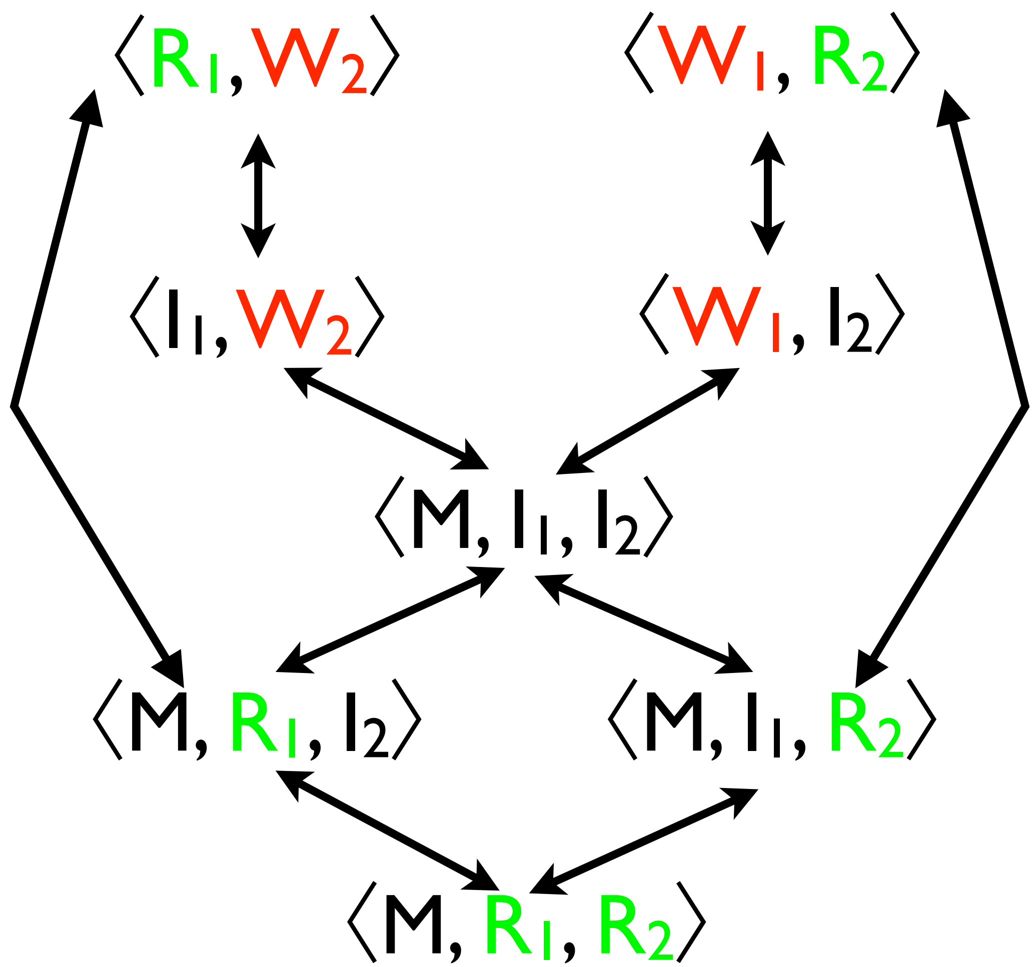
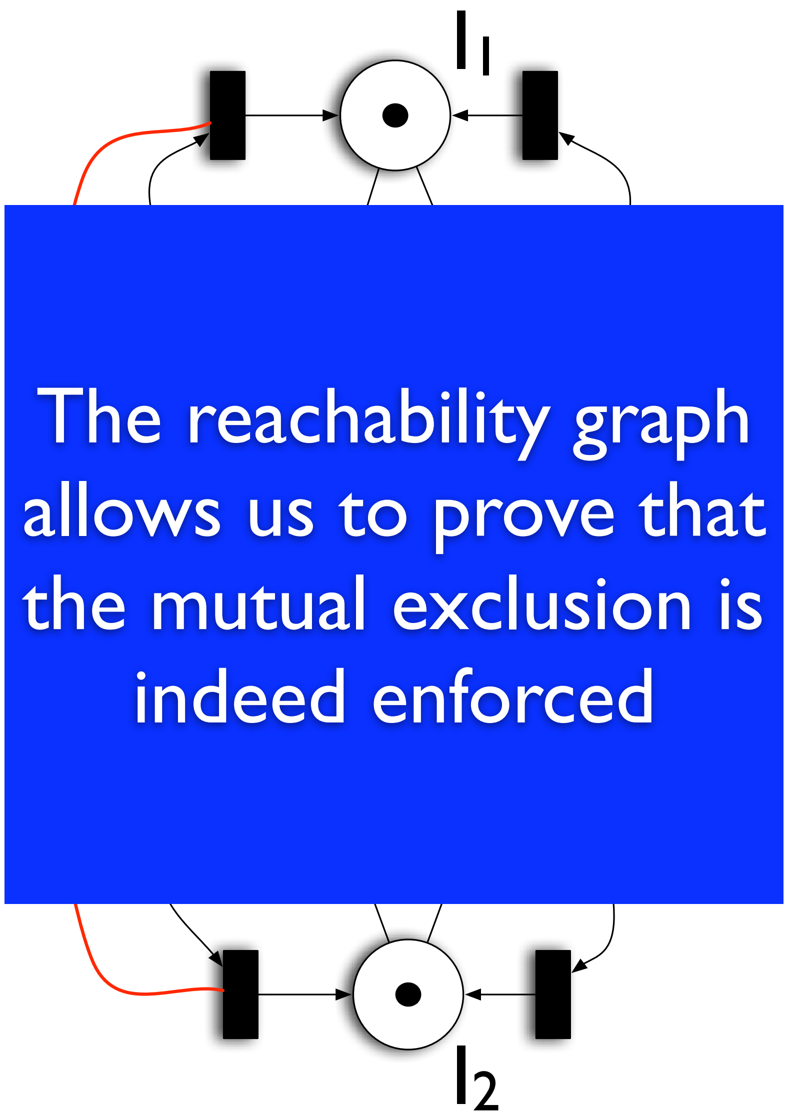
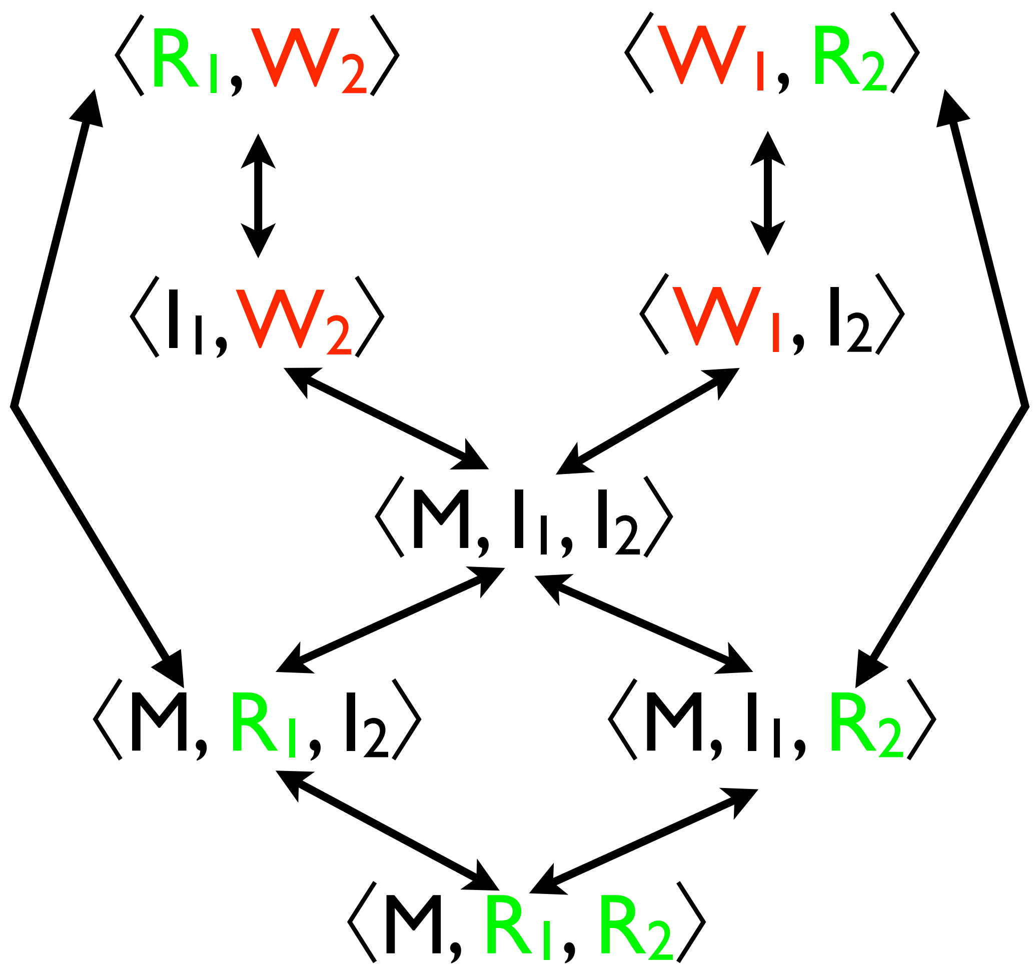


# Reachability graph




The reachability graph allows us to prove that the mutual exclusion is indeed enforced



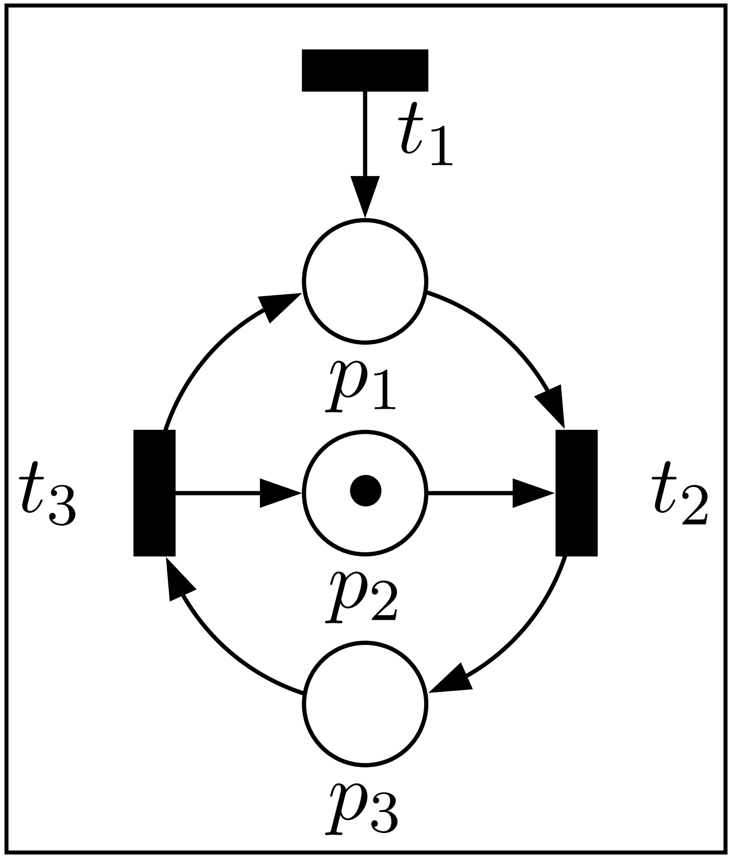
# Reachability graph

- The reachability graph of a PN contains all the necessary information to decide:
  - boundedness
  - place boundedness
  - semi-liveness
  - ...

# Reachability graph

- Unfortunately...

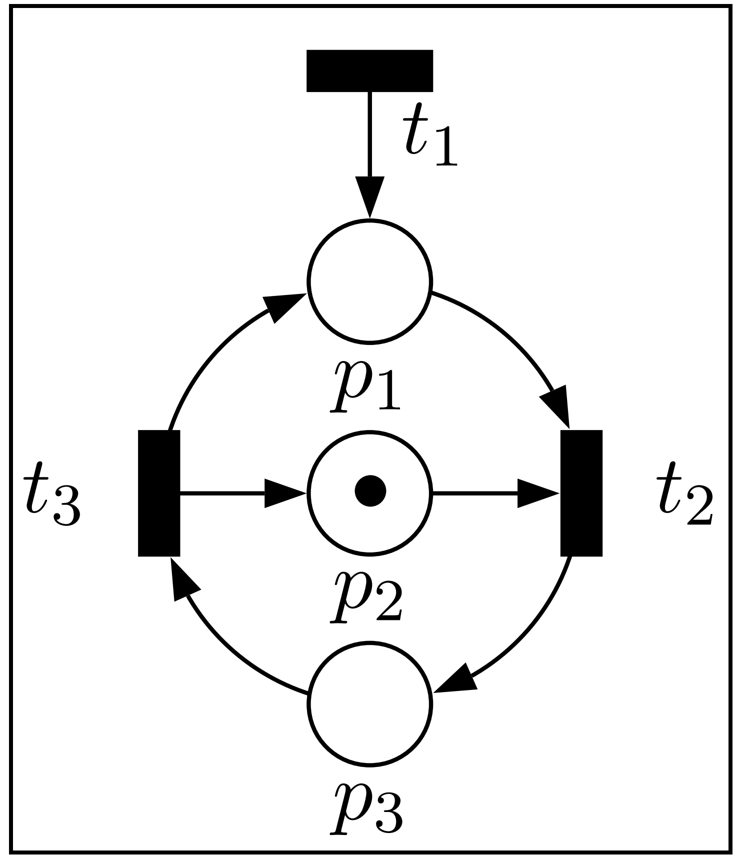
  $\langle p_2 \rangle$



# Reachability graph

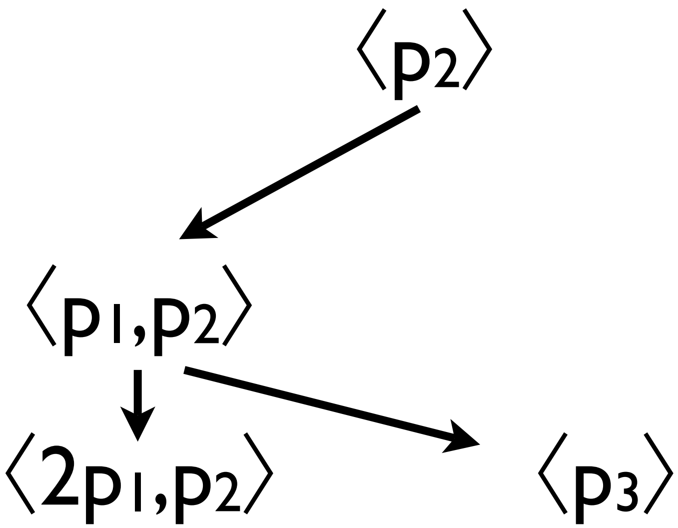
- Unfortunately...

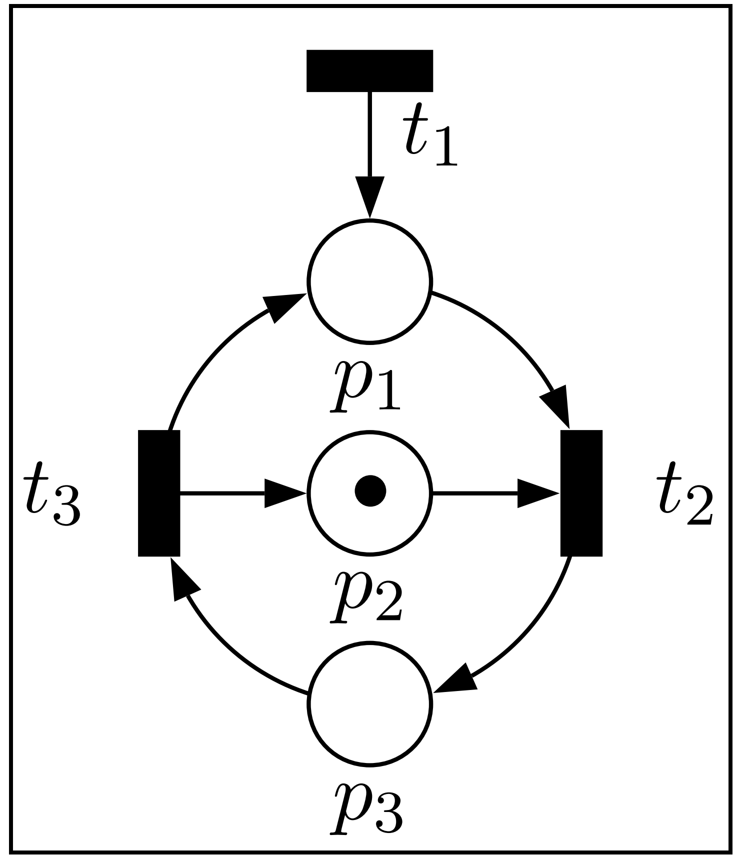
$$\langle p_2 \rangle \longrightarrow \langle p_1, p_2 \rangle$$



# Reachability graph

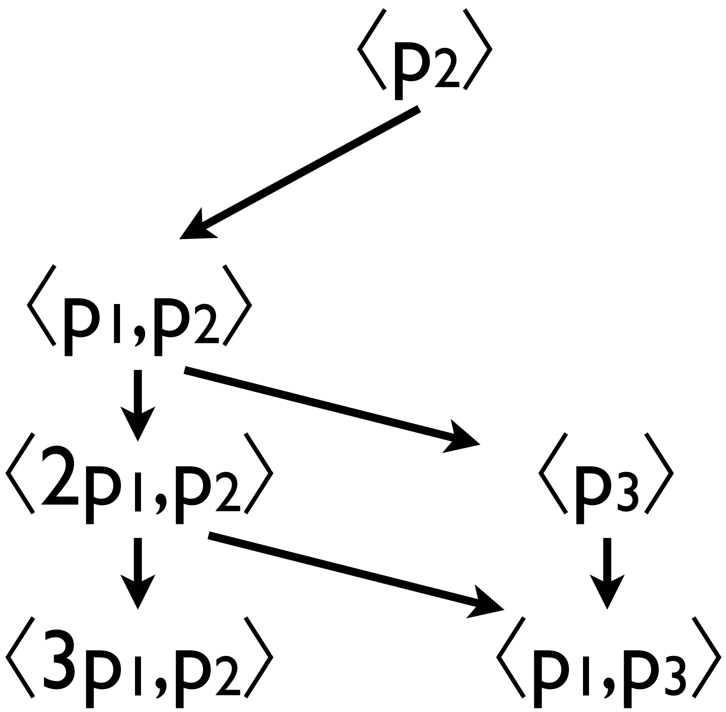
- Unfortunately...

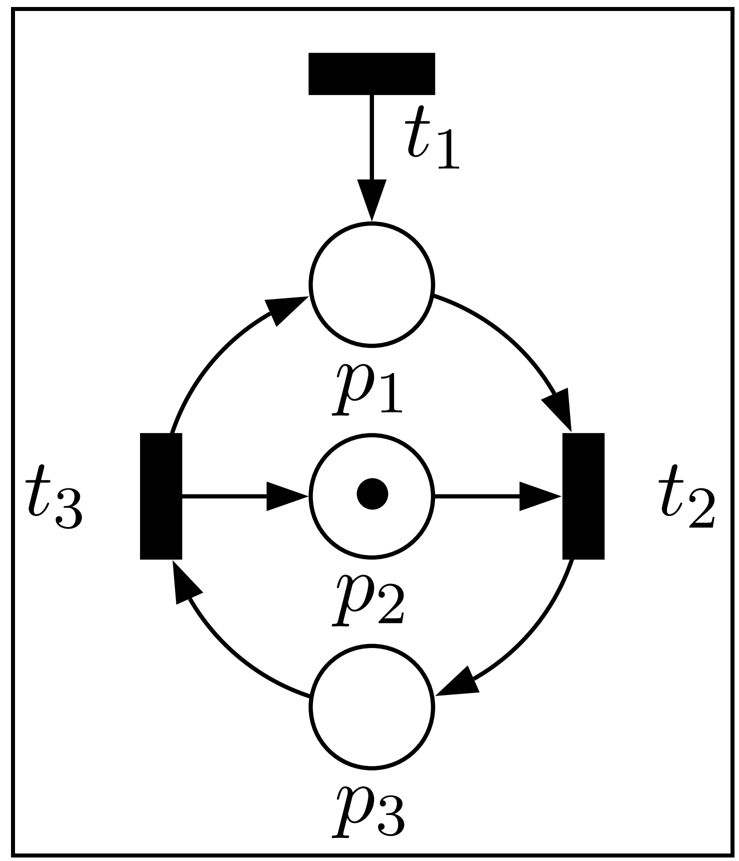


# Reachability graph

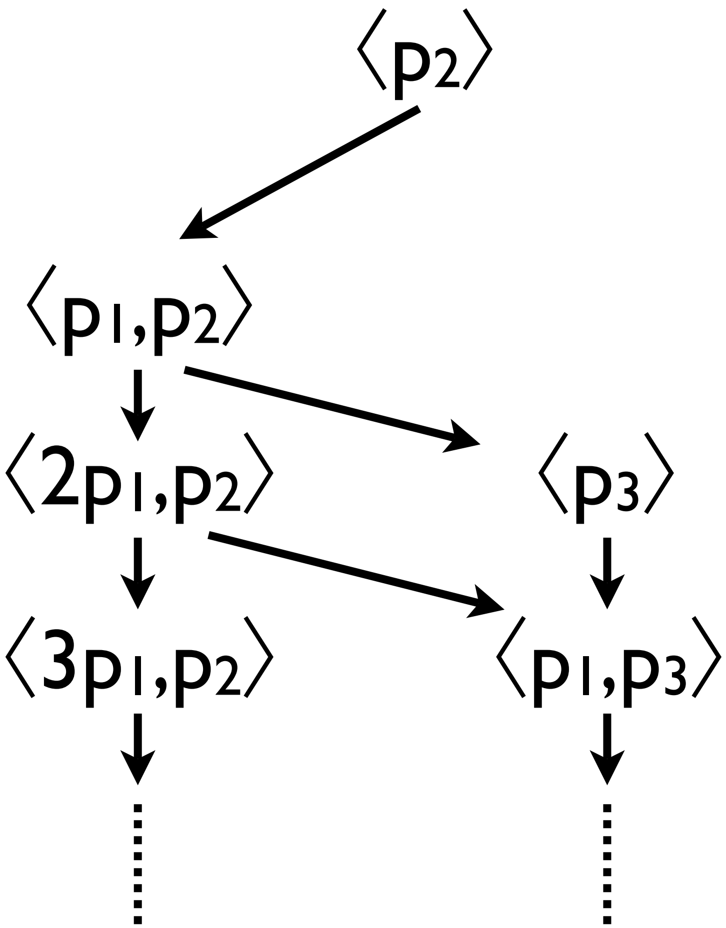
- Unfortunately...

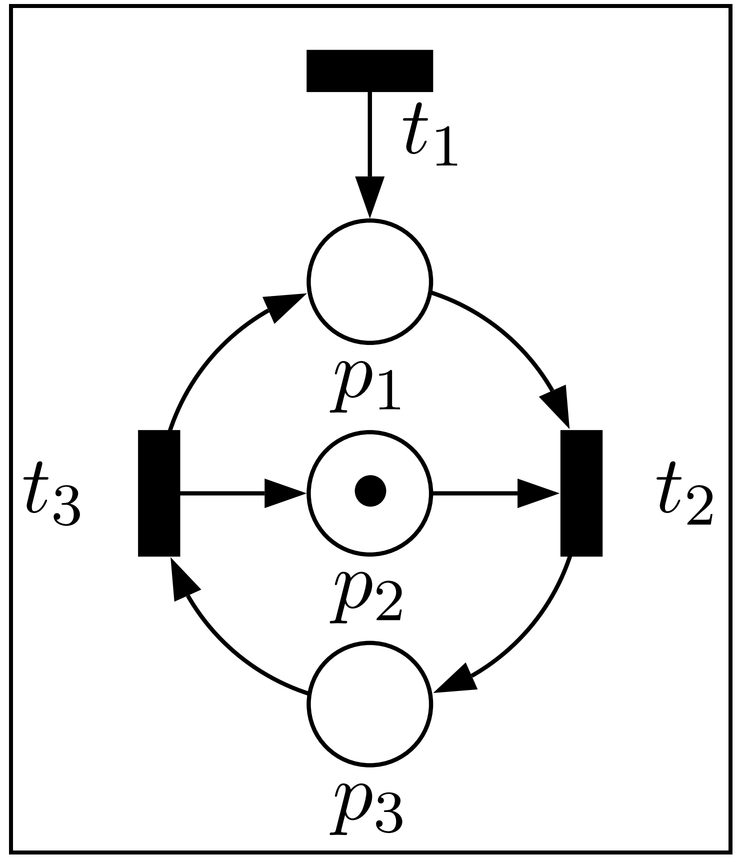


# Reachability graph

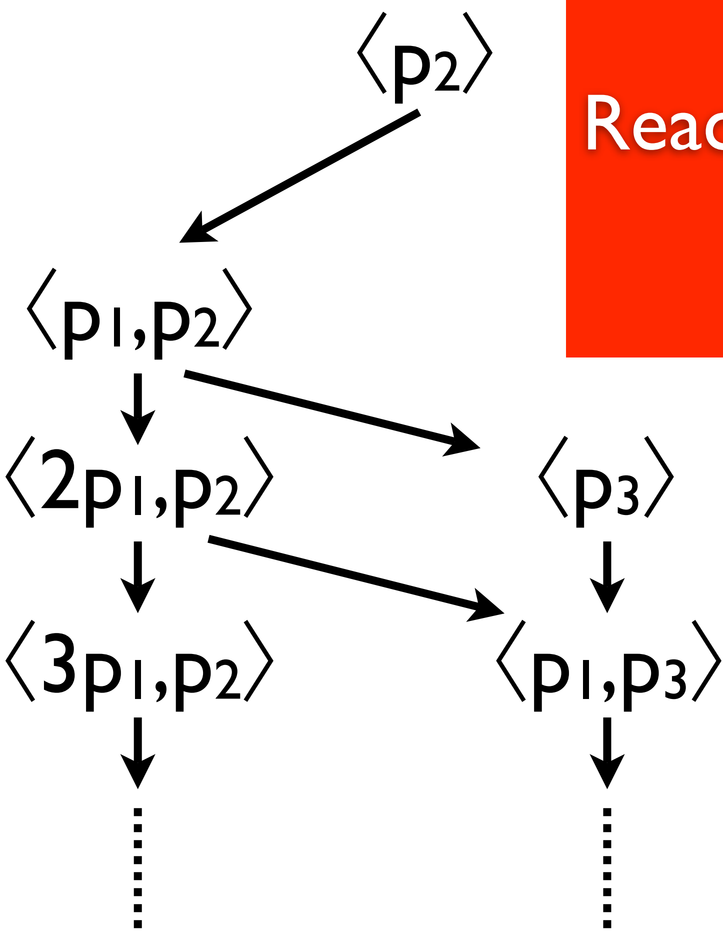
- Unfortunately...



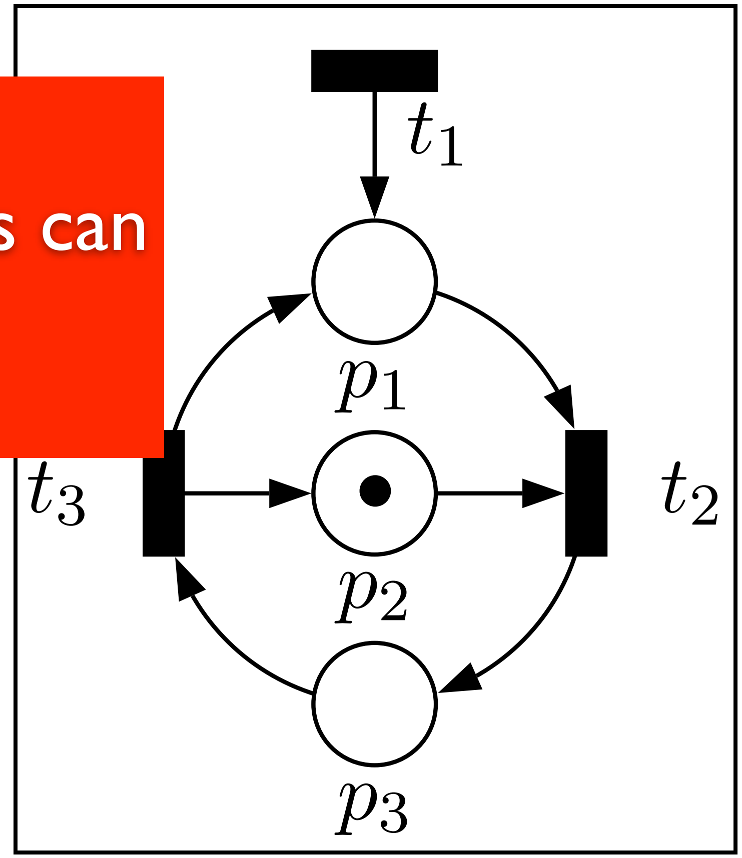



# Reachability graph

- Unfortunately...




Reachability graphs can be infinite



# The hard stuff...

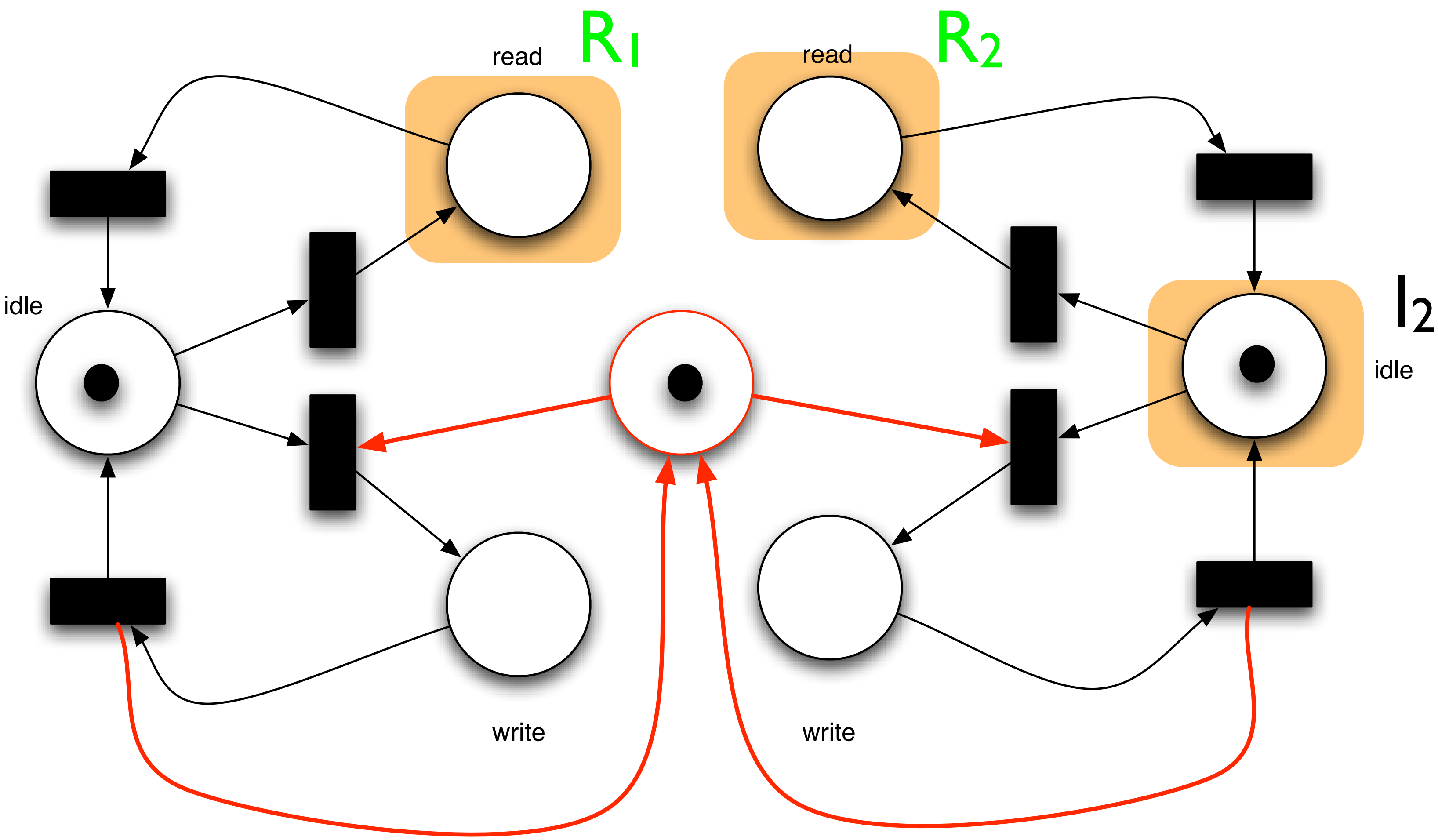
- The main difficulty in analysing Petri nets is due to the possibly infinite number of reachable markings.
  - We have to find techniques to deal with this infinite set.

# The hard stuff...

- Remark: finite doesn't mean easy
  - The set of reachable markings of a bounded net can be huge !
- Efficient techniques to deal with bounded nets have been developped.
  - e.g.: net unfoldings

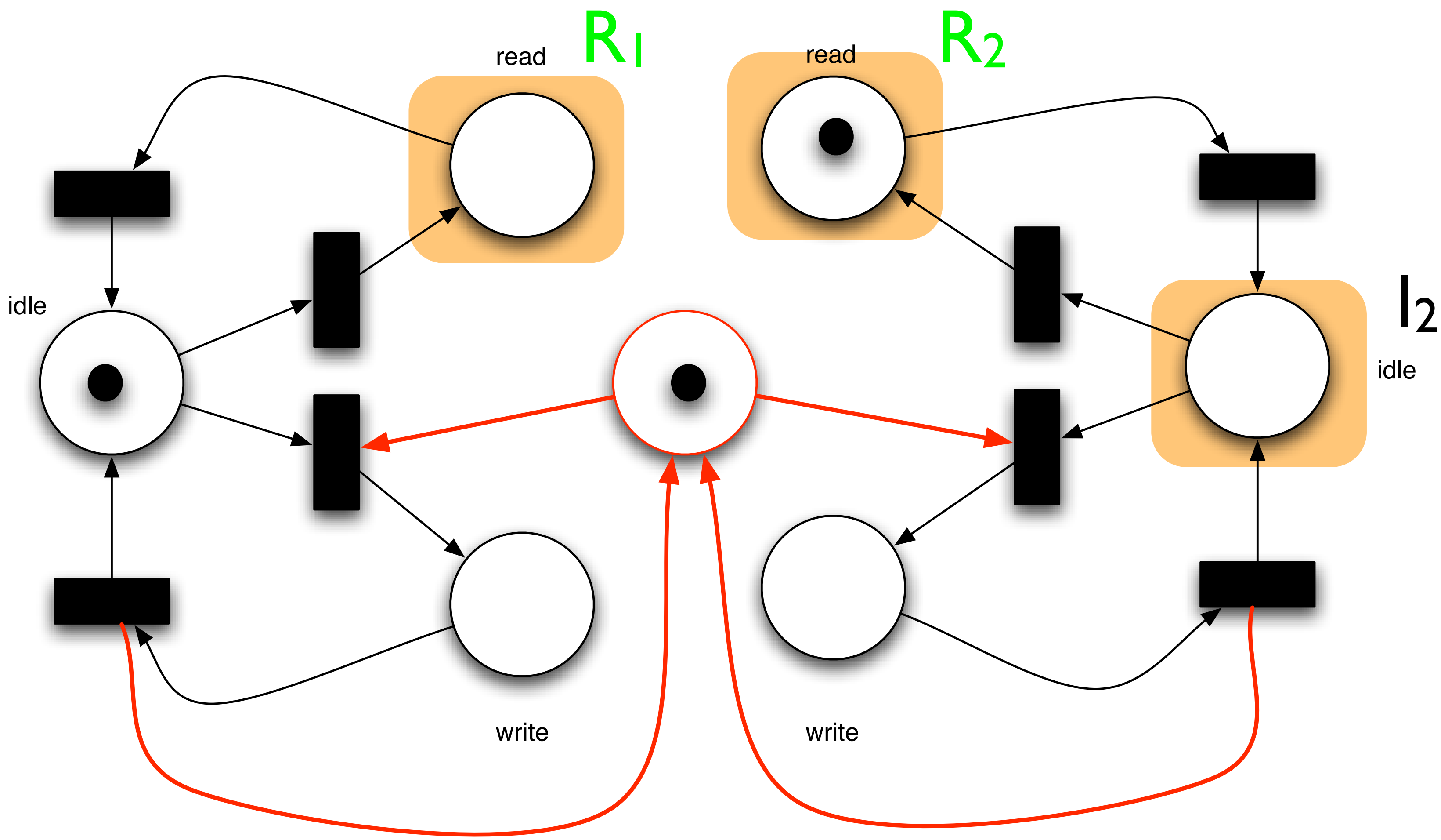
# Place invariants

# Place Invariants




$$m(R_1) + m(R_2) + m(I_2) = 1$$

# Place Invariants

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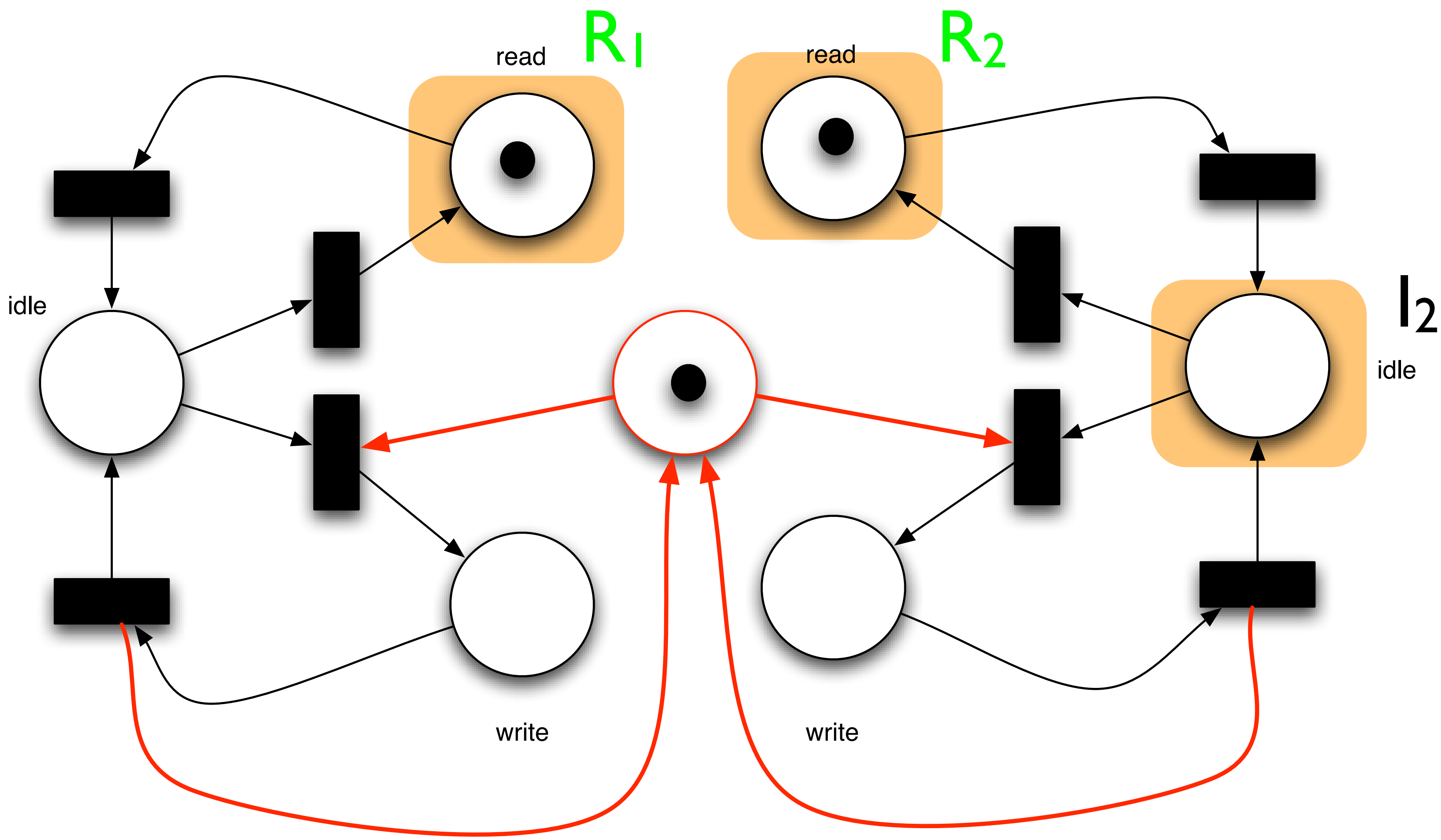
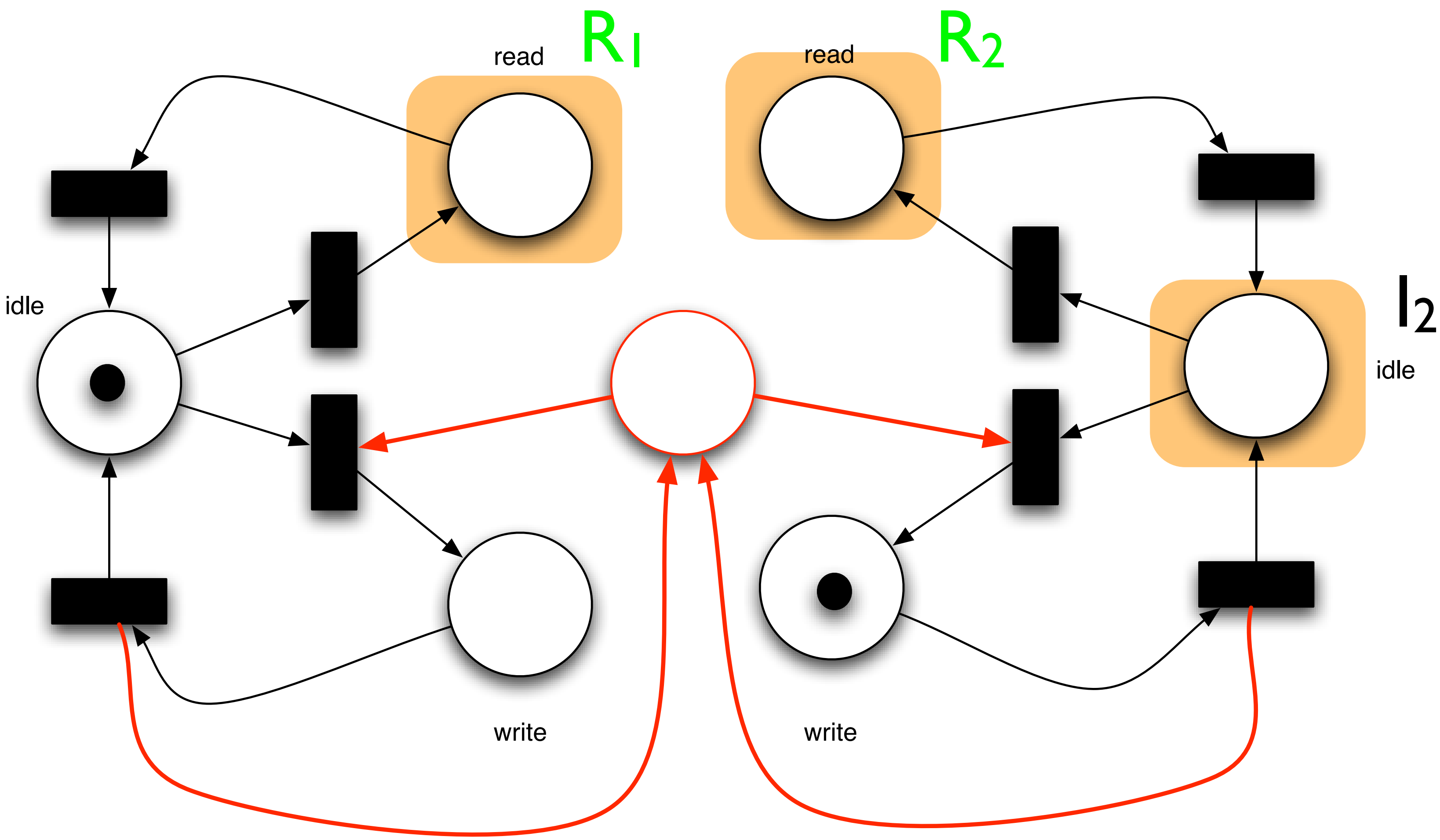
# Place Invariants



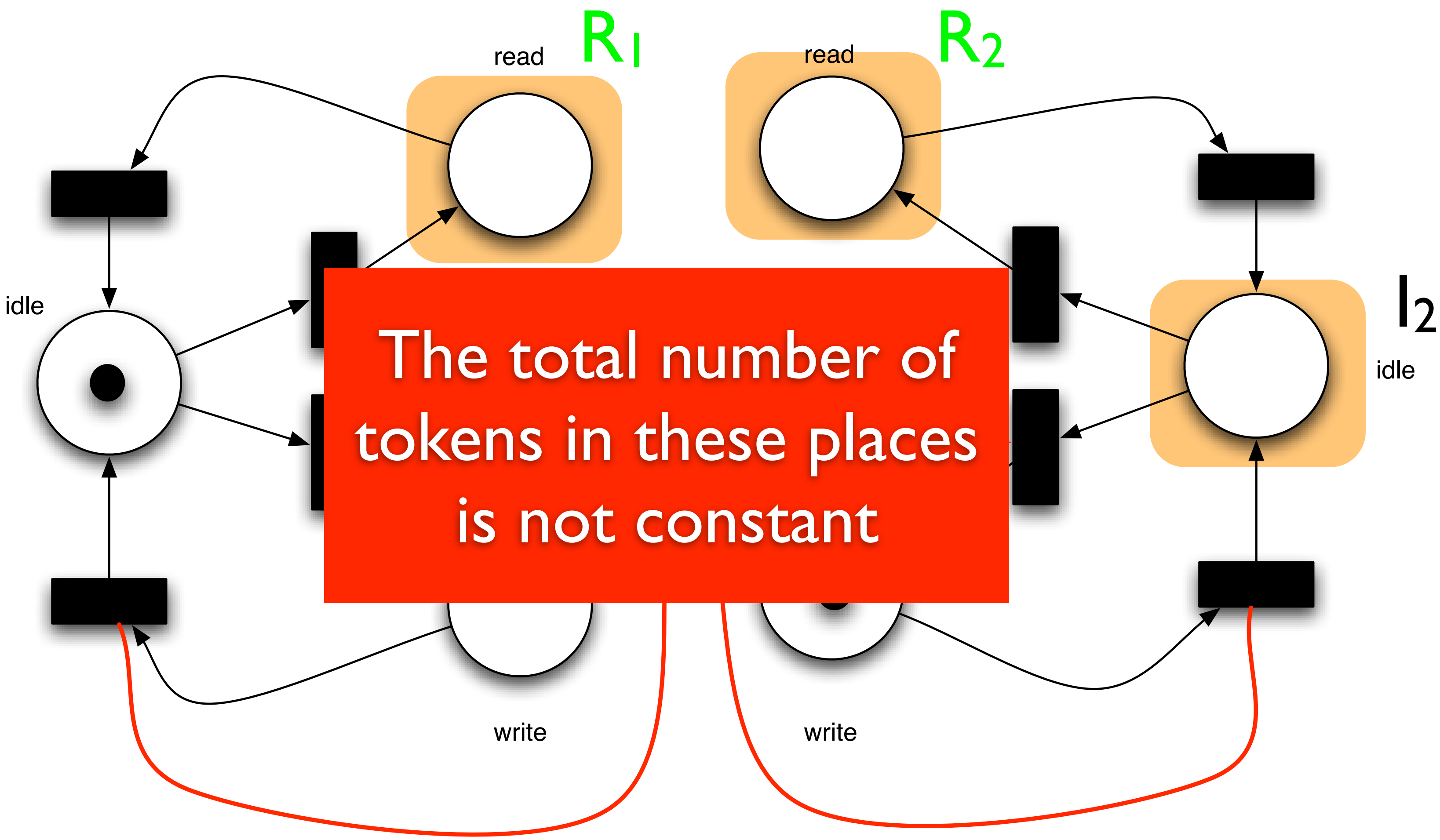

$$m(R_1) + m(R_2) + m(I_2) = 2$$

# Place Invariants



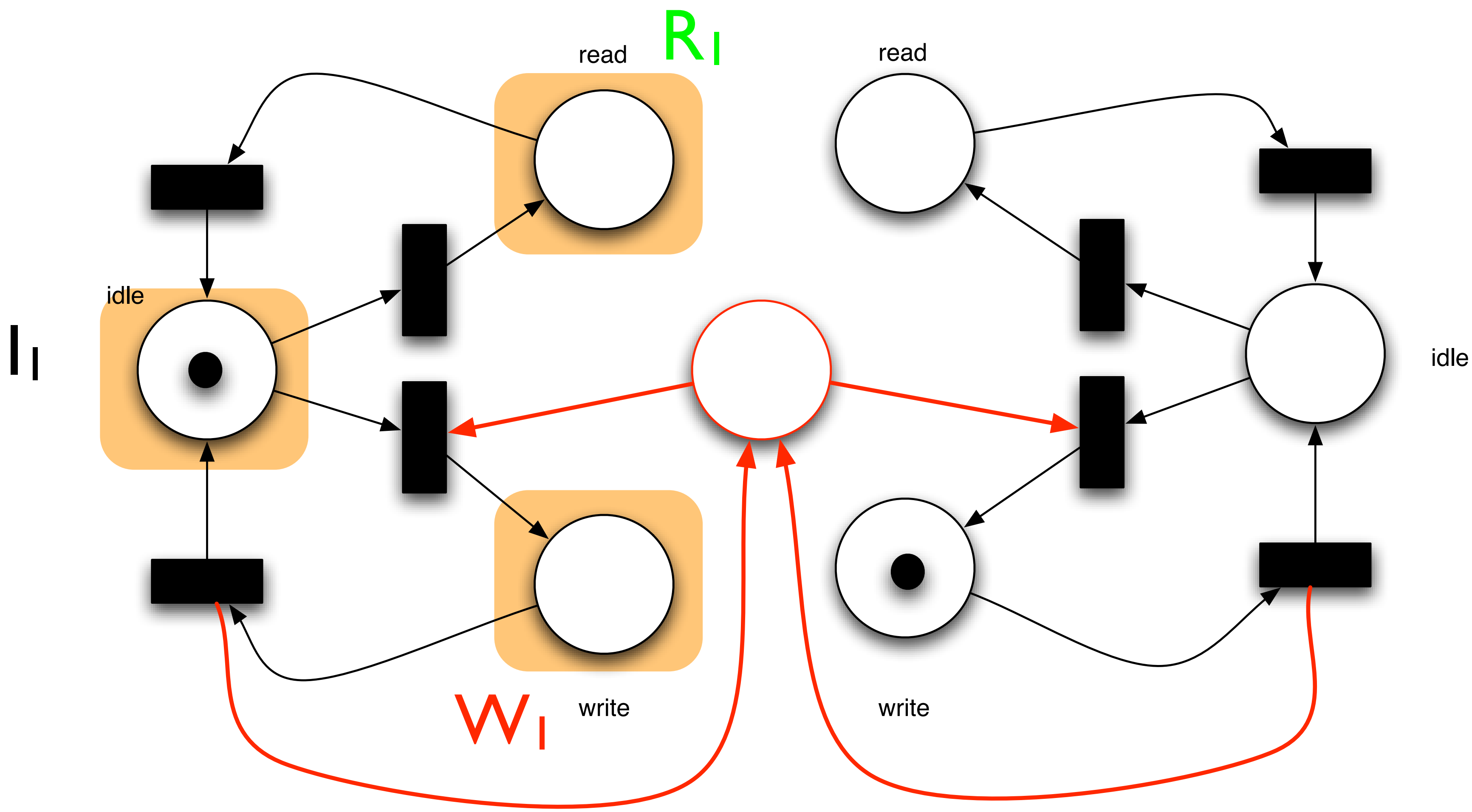

$$m(R_1) + m(R_2) + m(I_2) = 0$$

# Place Invariants



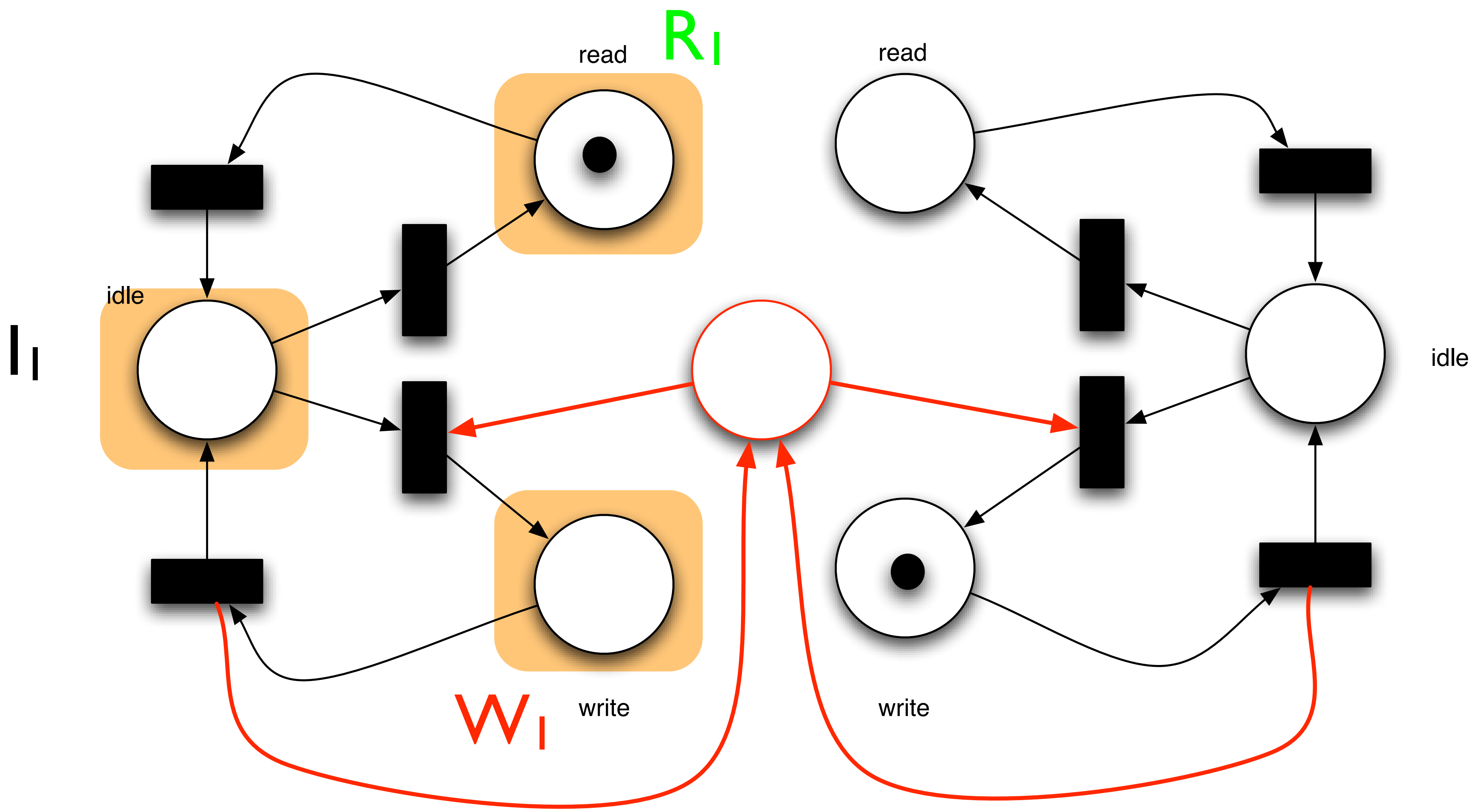

$$m(R_1) + m(R_2) + m(I_2) = 0$$

# Place Invariants




$$m(R_1) + m(W_1) + m(I_1) = 1$$

# Place Invariants

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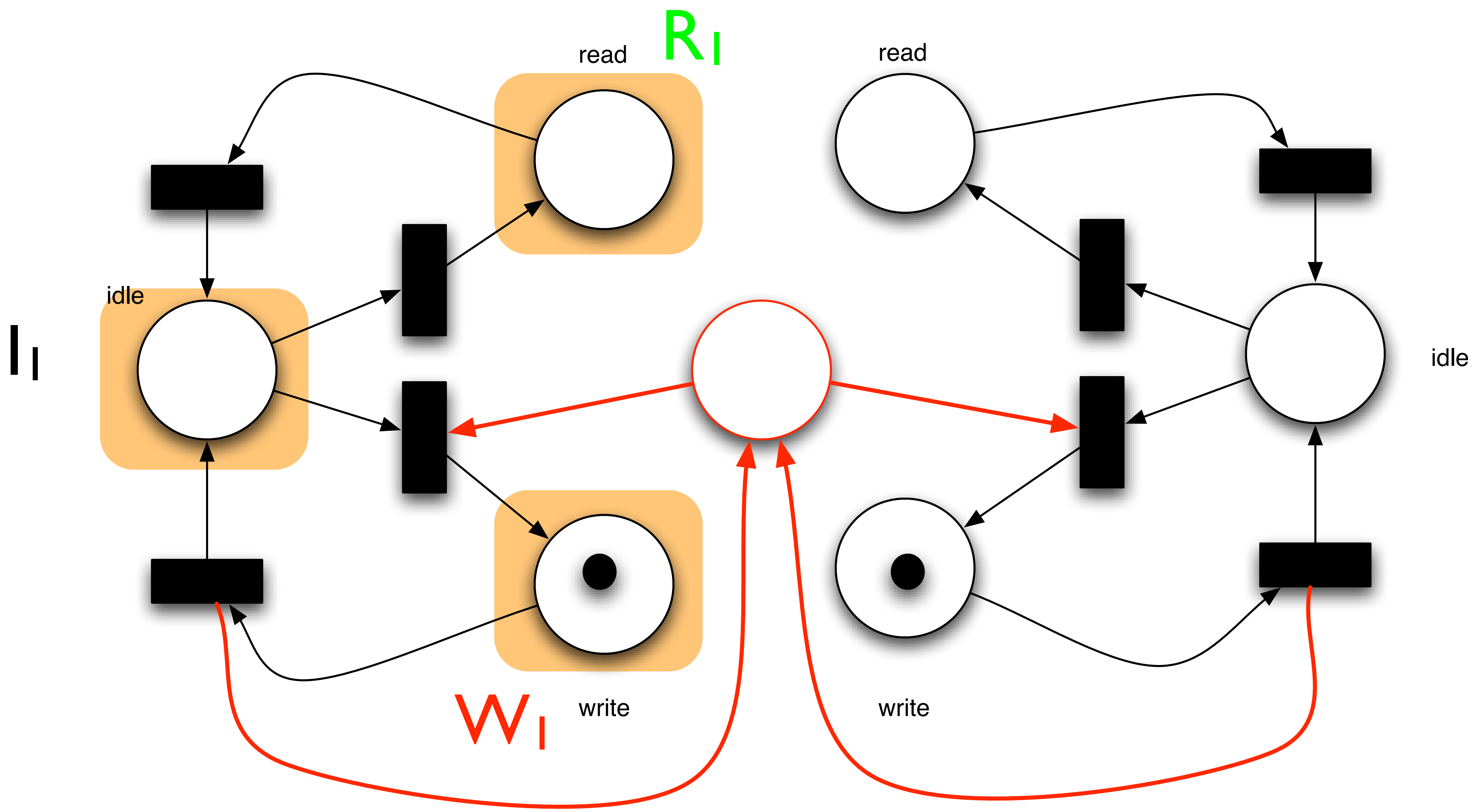
# Place Invariants

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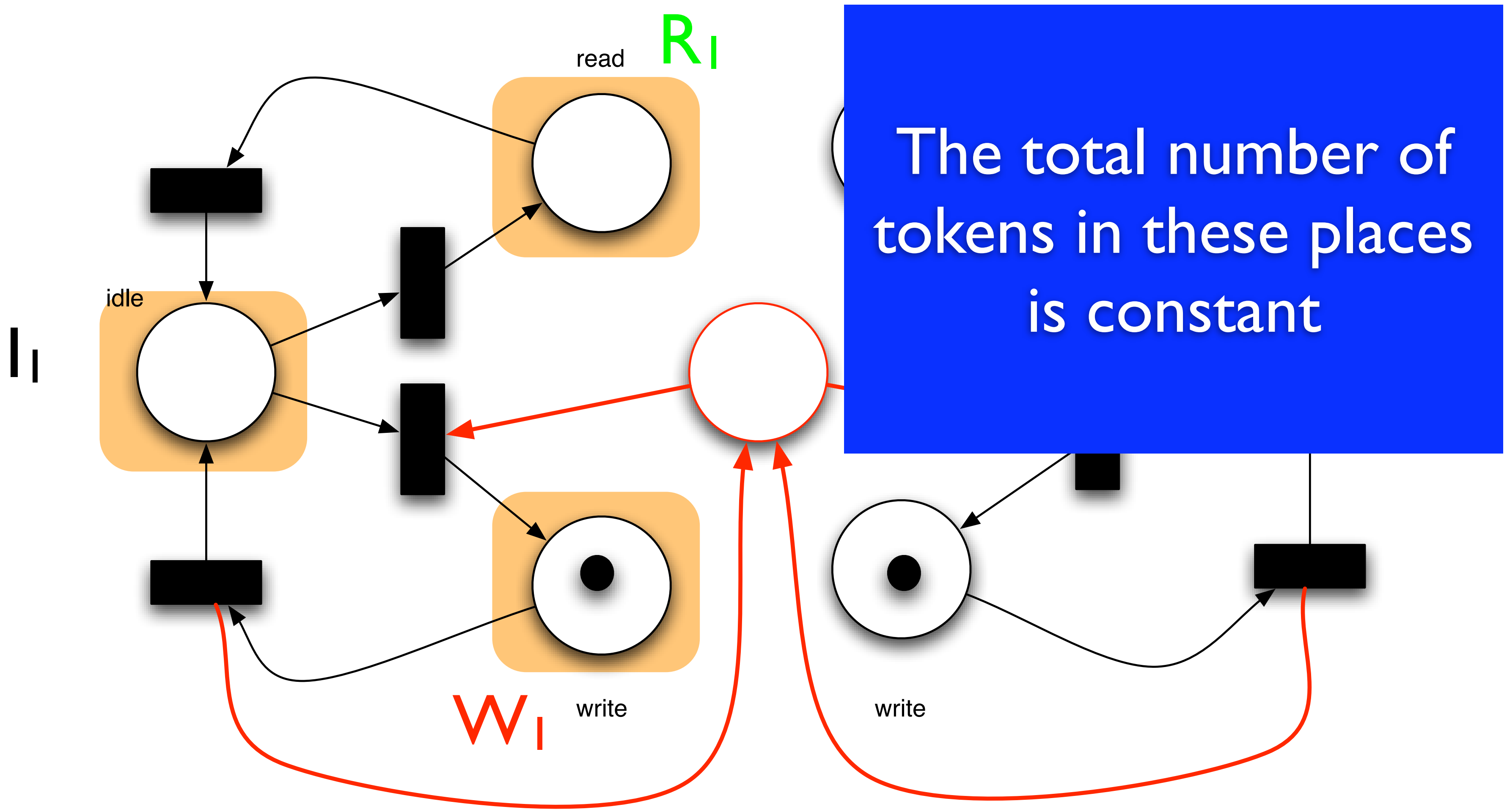


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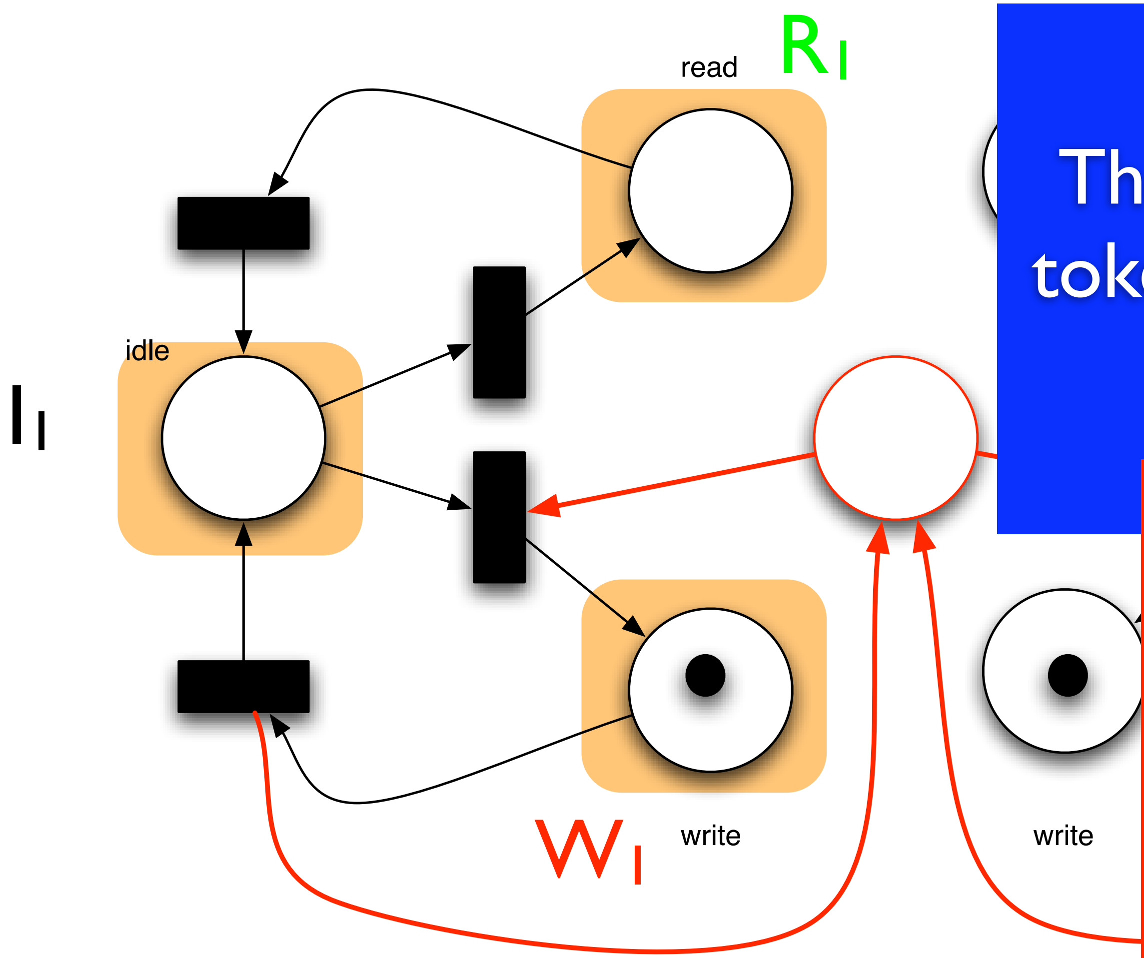
# Place Invariants

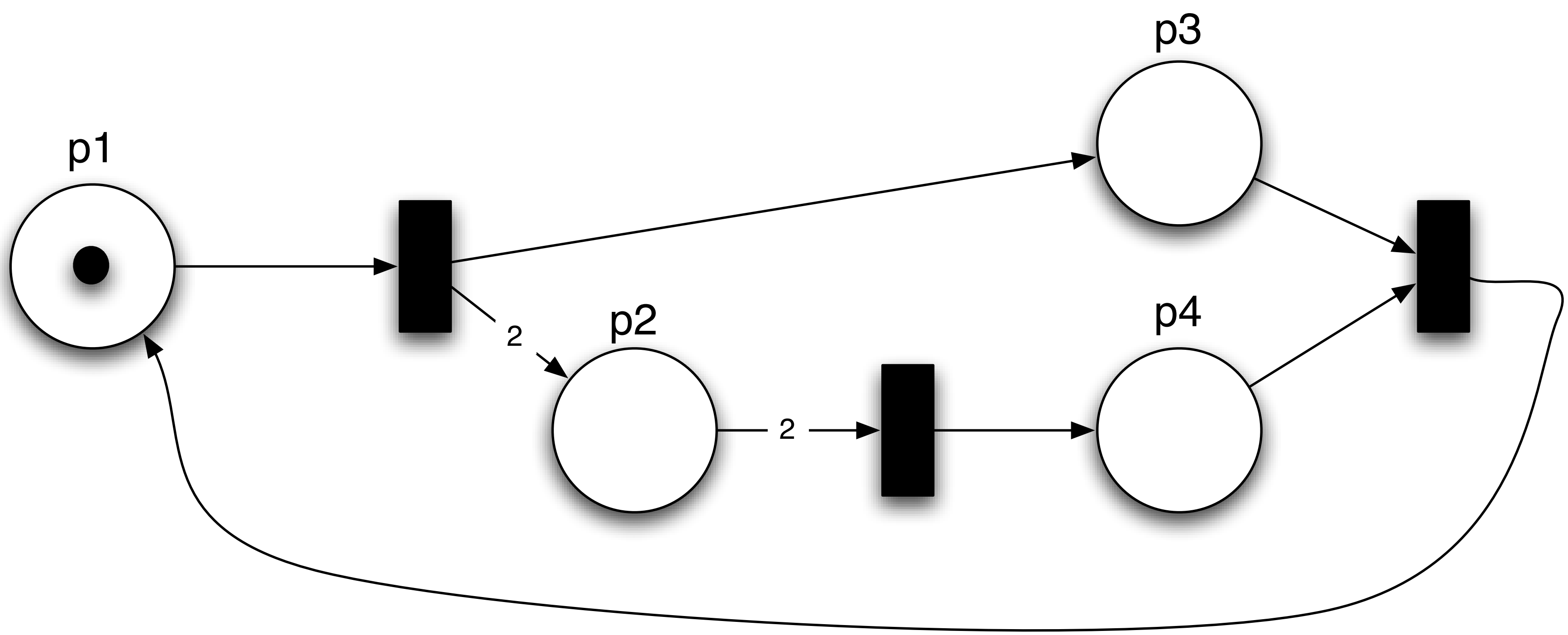



The total number of tokens in these places is constant

This provides meaningful information about the system: a process is either idle, or reading or writing

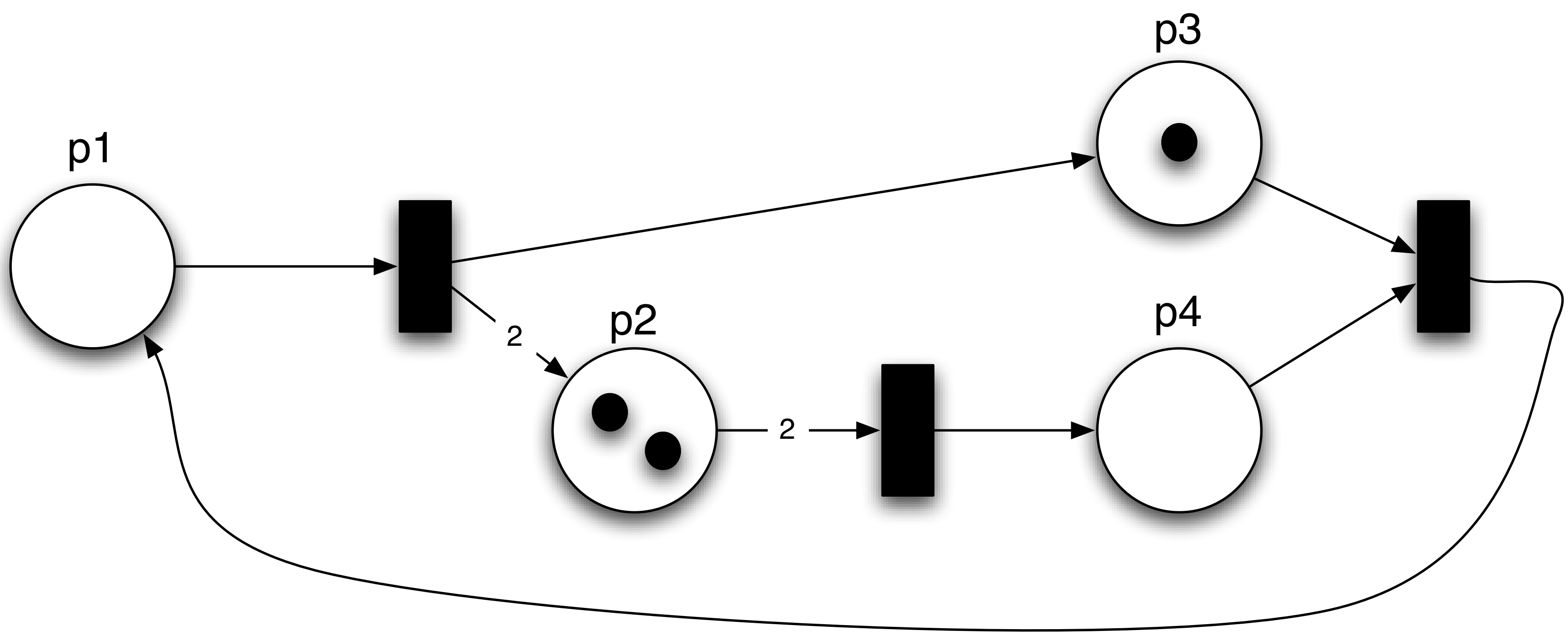
$$m(R_1) + m(W_1) + m(I_1) = 1$$

# Place Invariants



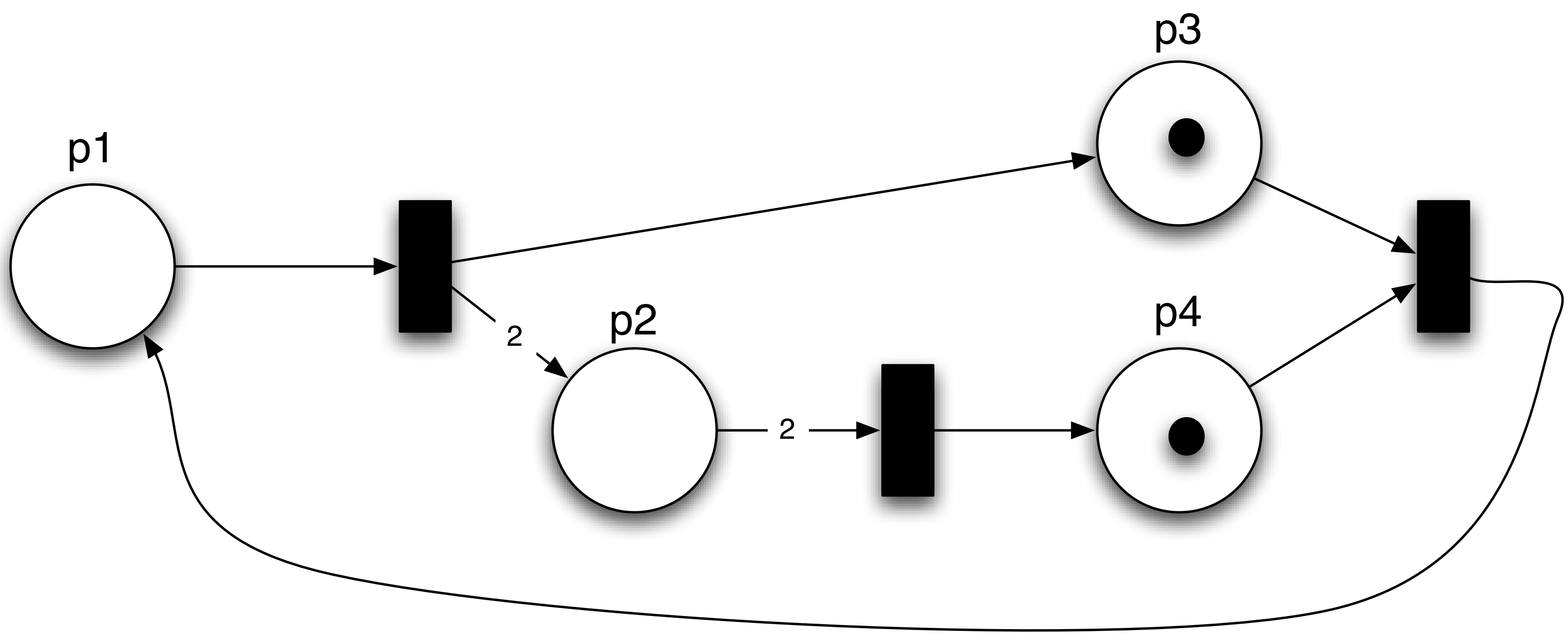

$$m(p_1) + m(p_2) + m(p_3) + m(p_4) = 1$$

# Place Invariants



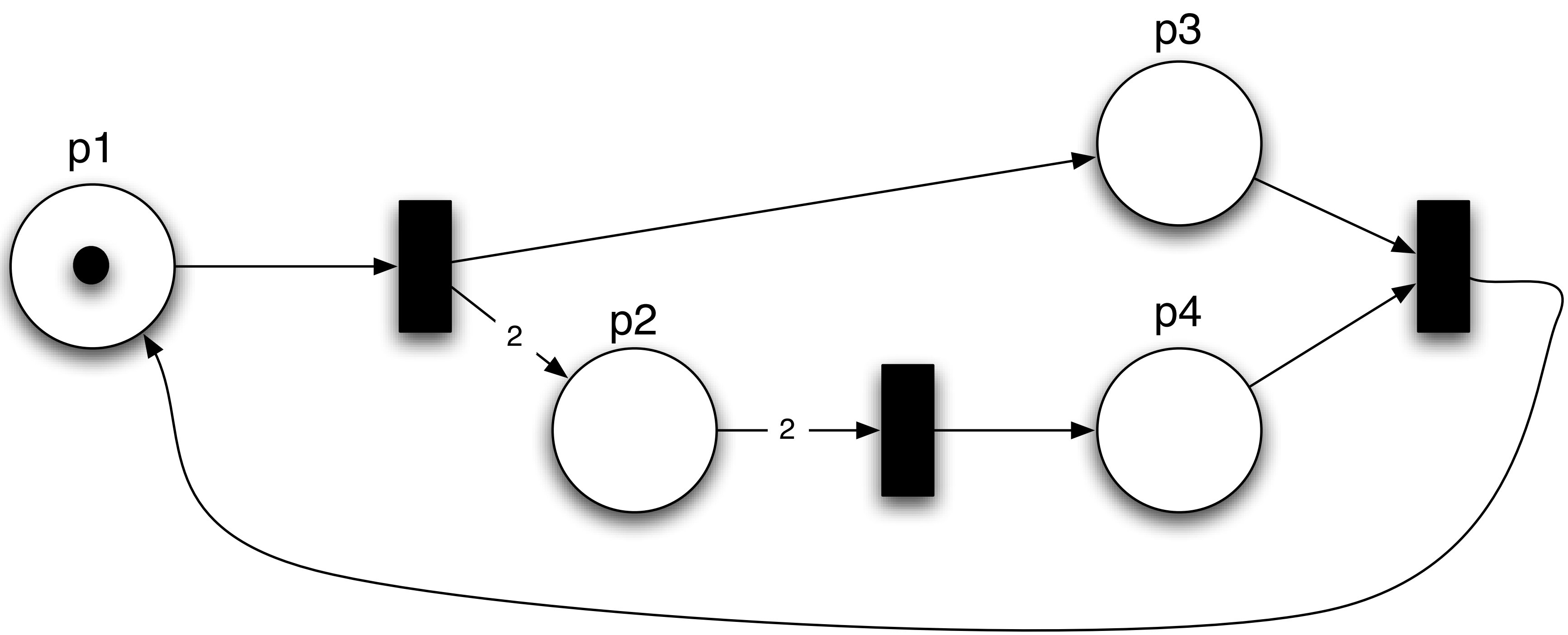

$$m(p_1) + m(p_2) + m(p_3) + m(p_4) = 3$$

# Place Invariants



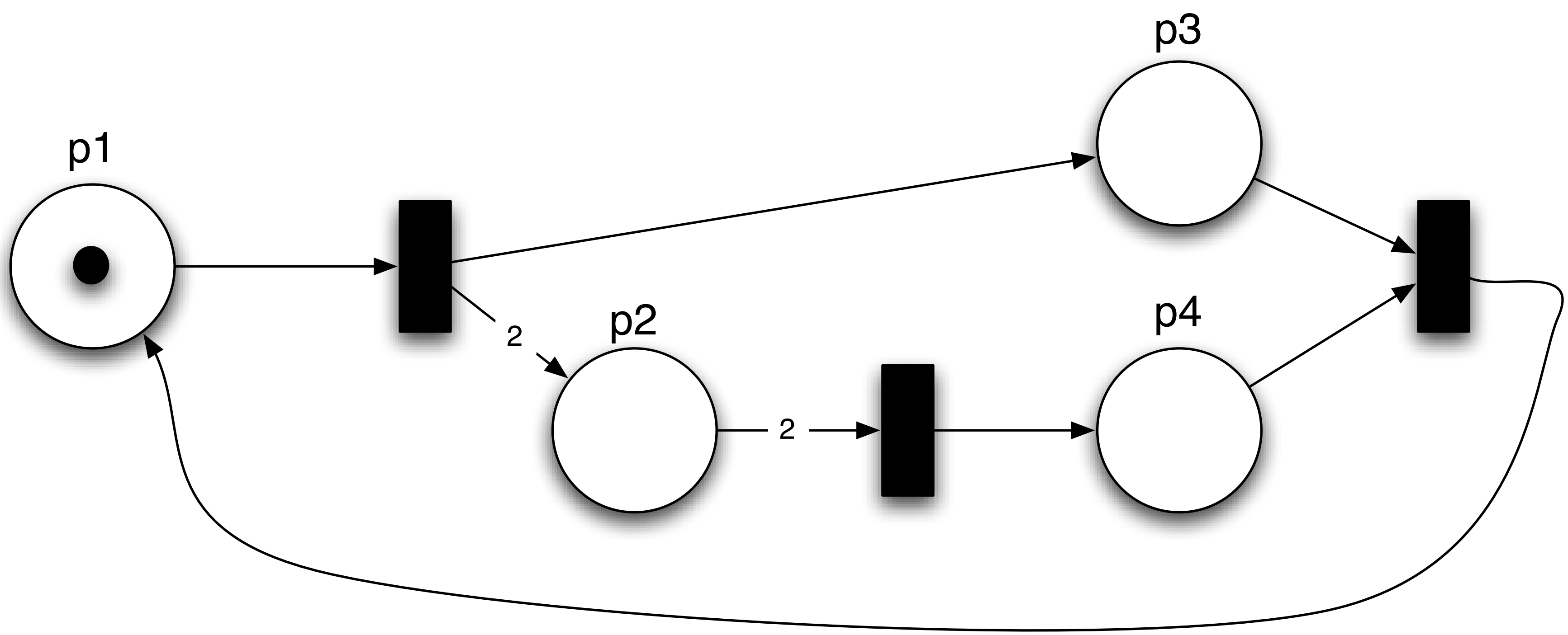

$$m(p_1) + m(p_2) + m(p_3) + m(p_4) = 2$$

# Place Invariants




$$m(p_1) + m(p_2) + m(p_3) + m(p_4) = 1$$

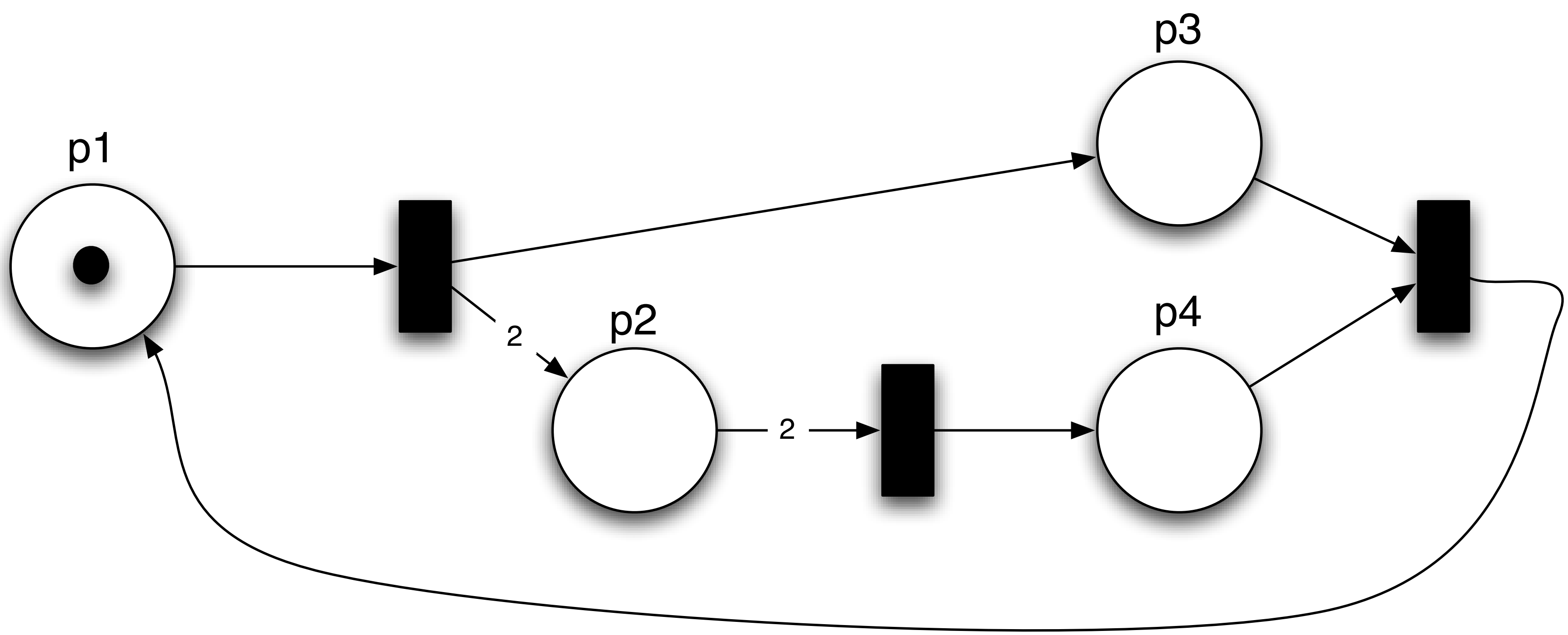
# Place Invariants




The total number of tokens in these places is not constant

$+ m(p_3) + m(p_4) = 1$

# Place Invariants




The total number of tokens in these places is not constant

+ m(p

In some sense, tokens in $p_1$ are heavier than those in $p_2$

# Place Invariants

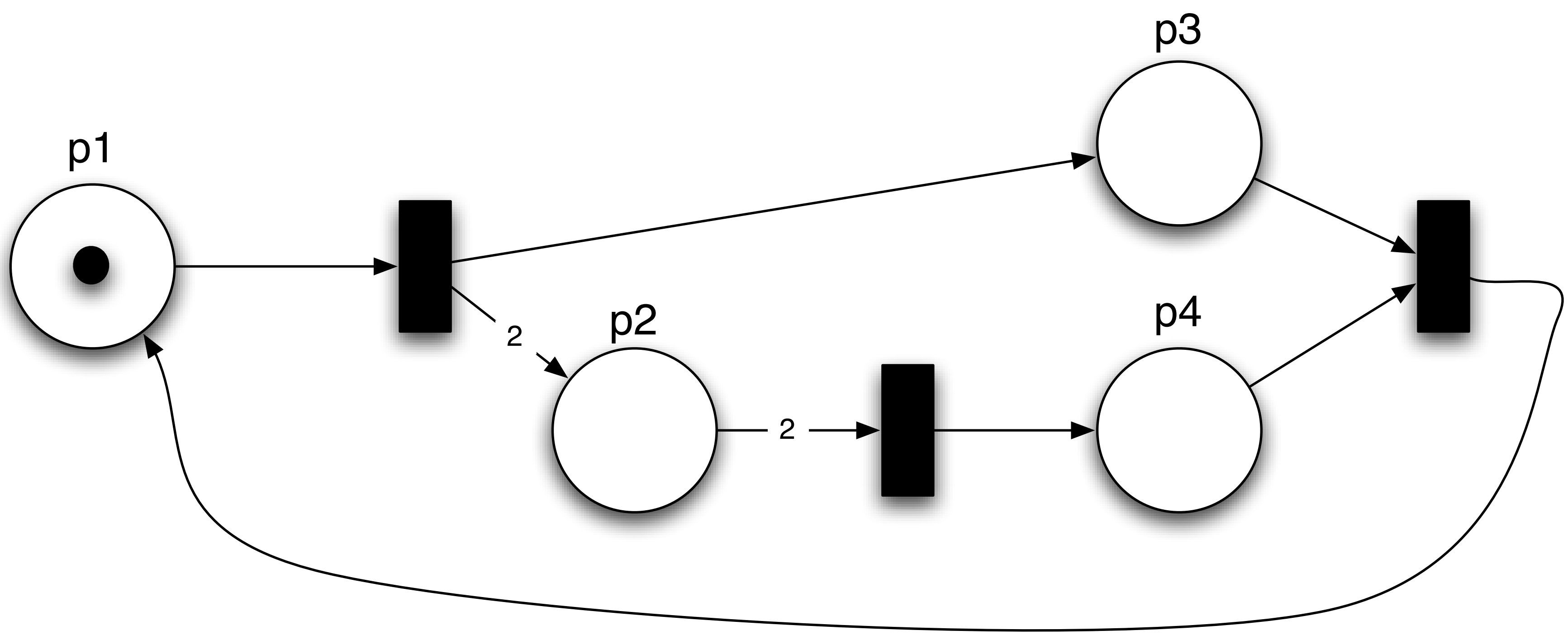

Let's add weights to the places !

The total number of tokens in these places is not constant

+ m(p

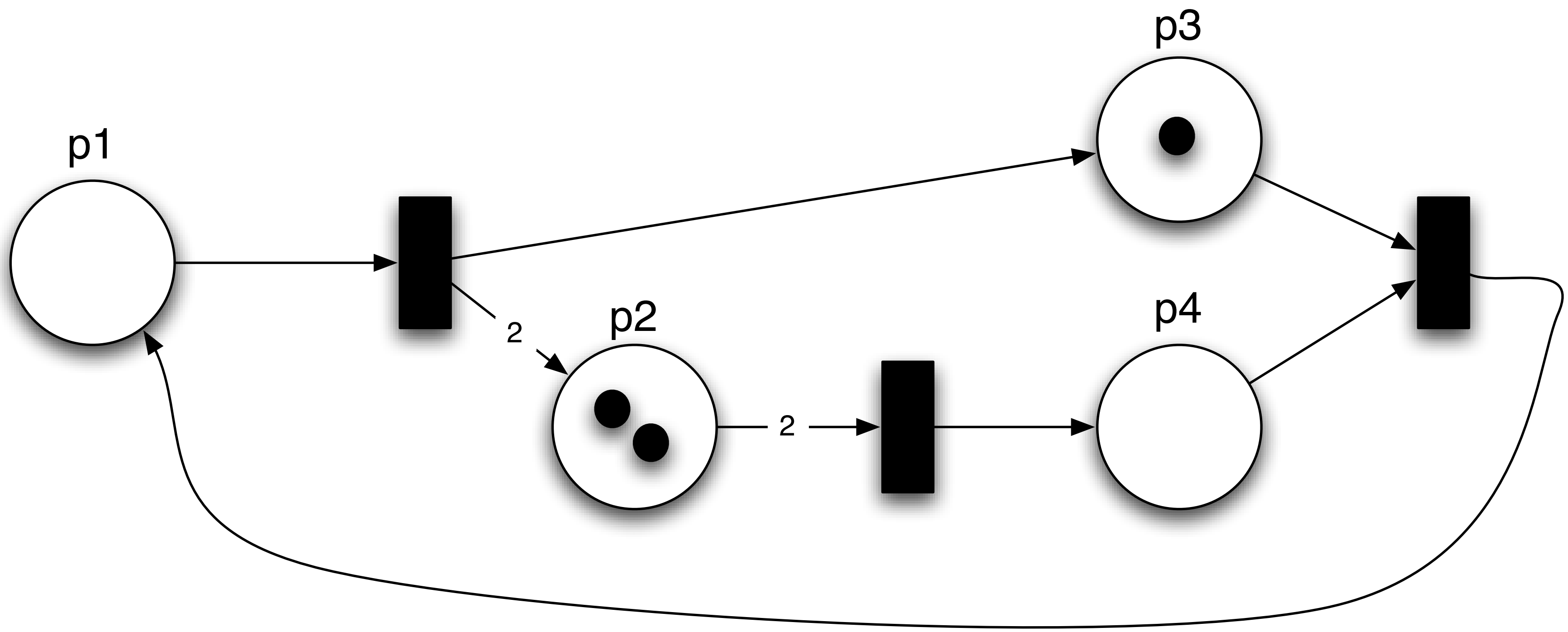
In some sense, tokens in $p_1$ are heavier than those in $p_2$

# Place Invariants



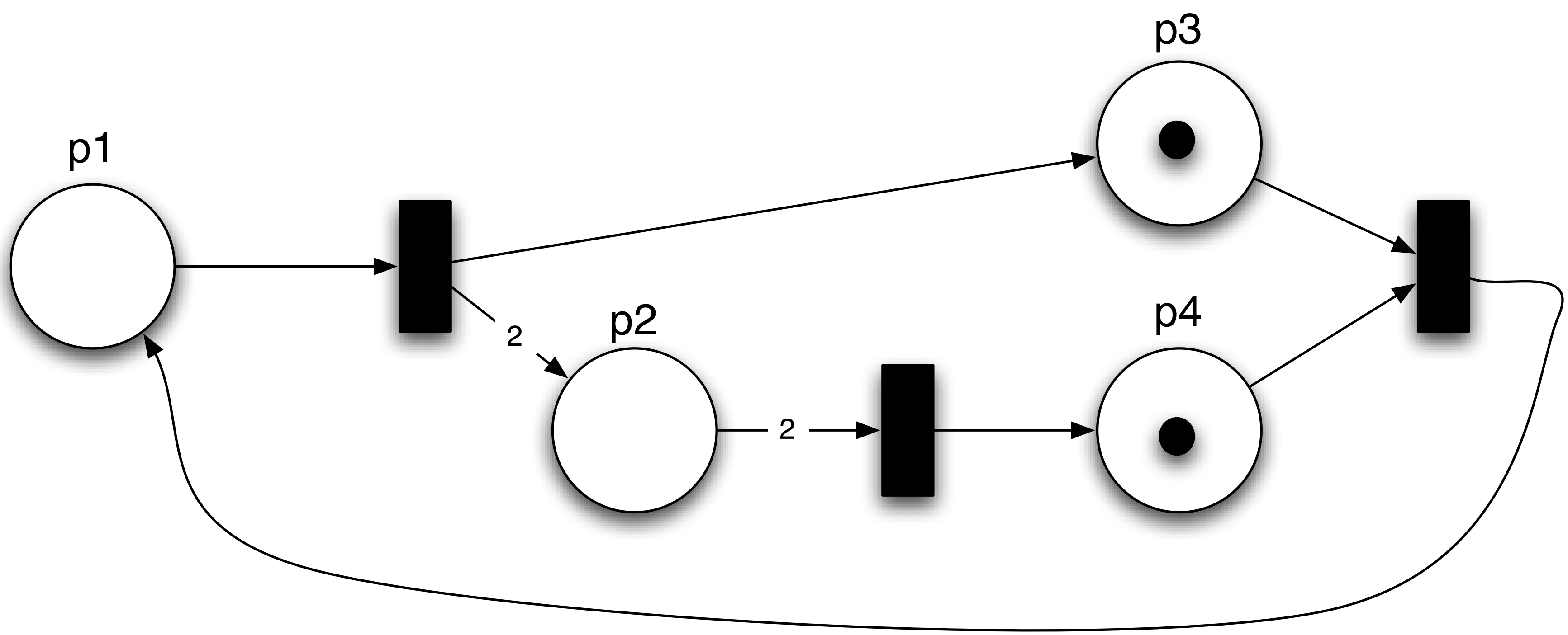

$$3\ m(p_1) + m(p_2) + m(p_3) + 2\ m(p_4) = 3$$

# Place Invariants




$$3\ m(p_1) + m(p_2) + m(p_3) + 2\ m(p_4) = 3$$

# Place Invariants




$$3\ m(p_1) + m(p_2) + m(p_3) + 2\ m(p_4) = 3$$

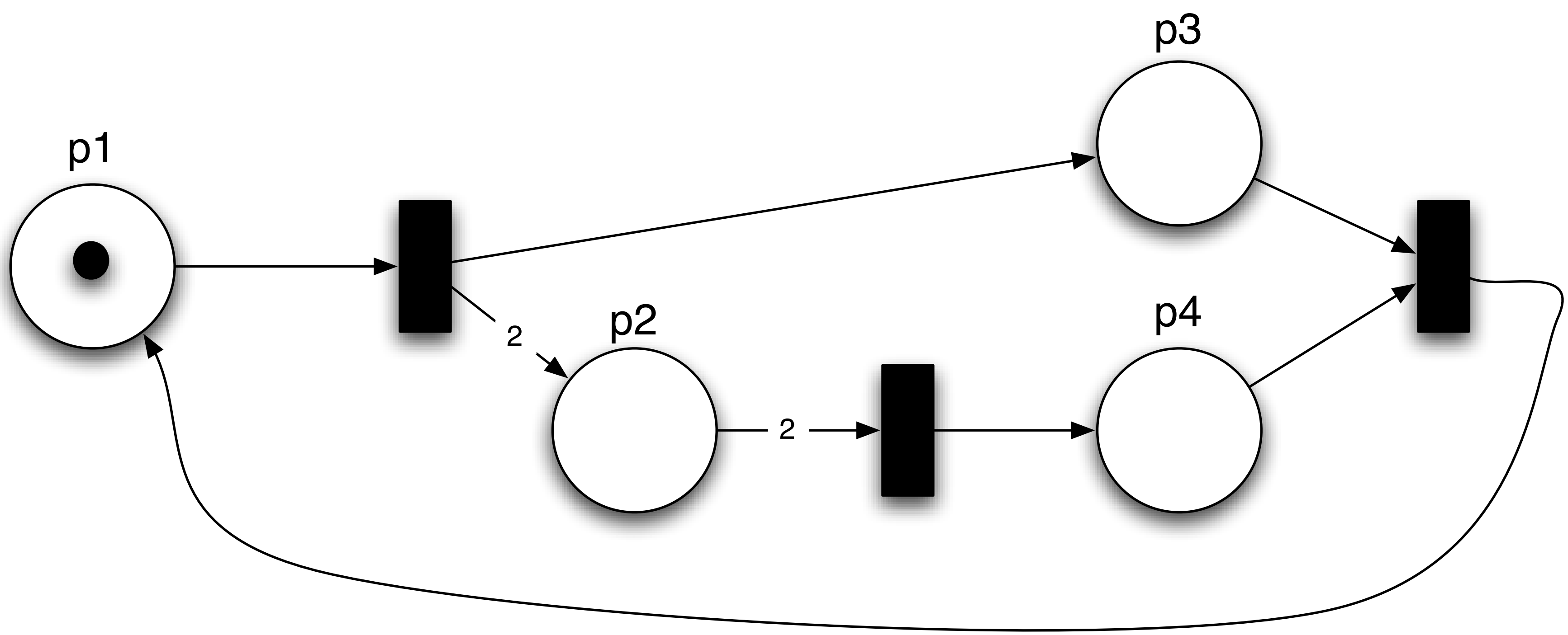
# Place invariant: Definition

- Definition: a place-invariant (or p-semiflow) is a vector i of natural numbers s.t. for any reachable marking m:

$$\sum_{p \in P} i(p) \times m(p) = \sum_{p \in P} i(p) \times m_0(p)$$

remark: there exists a trivial invariant i = $\langle 0, 0, .., 0 \rangle$

# Example: other invariants




$$m(p_1) + m(p_3) = 1$$

$$2\, m(p_1) + m(p_2) + 2\, m(p_4) = 2$$

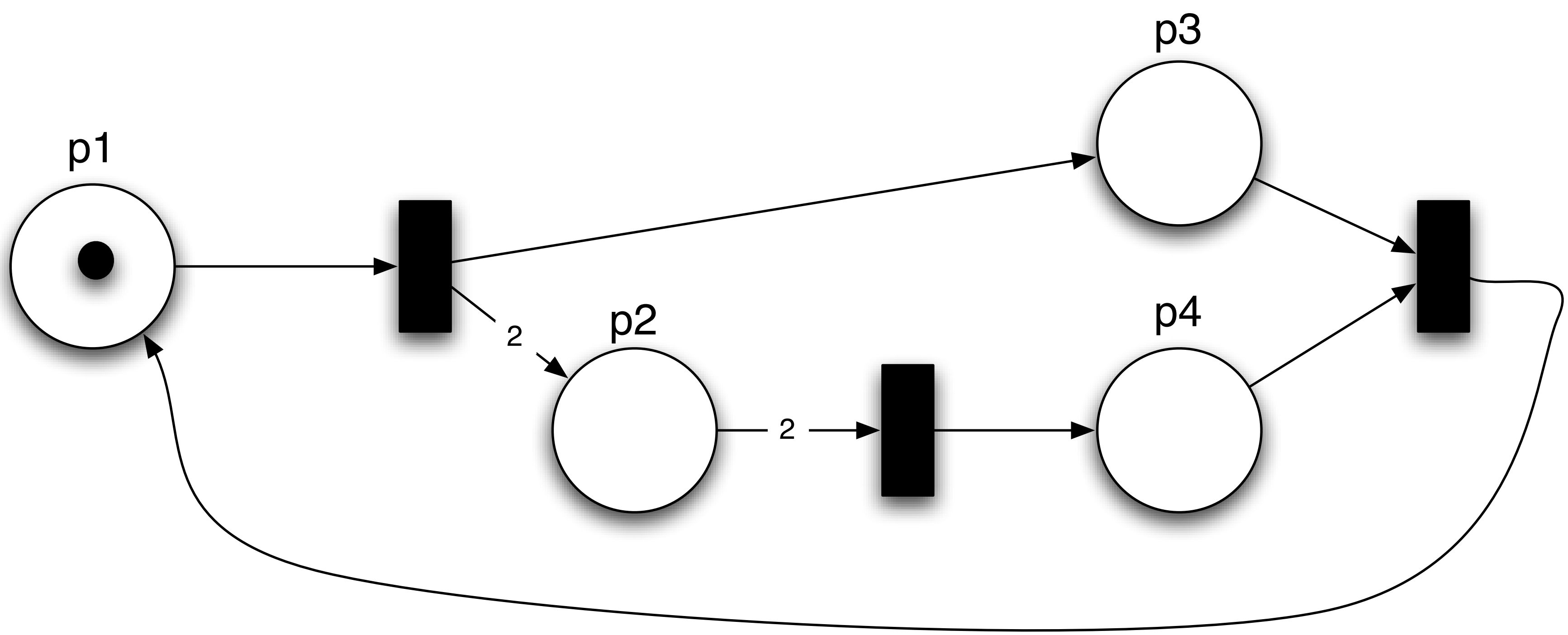
# Invariants as over-approximations

- A place-invariant expresses a constraint on the reachable markings.
  - If m is reachable and i is an invariant, then:

$$\sum_{p \in P} i(p) \times m(p) = \sum_{p \in P} i(p) \times m_0(p)$$

- The reverse is not true !

# Example




$$m(p_1) + m(p_3) = 1$$

is an invariant

but $\langle 1, 25, 0, 234 \rangle$ is not reachable

# Invariants as over-approximations

- Theorem: For any Petri net N:

$$\mathrm{Reach}(N)$$

$$\subseteq$$

$$\{m \mid m \text{ respects some invariant of } N\}$$

# Invariants as over-approximations

- Theorem: For any Petri net N:

$$\mathrm{Reach}(N) \subseteq \{m \mid m \text{ respects some invariant of } N\}$$

This set overapproximates the reachable markings

# Invariants as over-approximations

- Theorem: For any Petri net N:

Reach(N)

$\subseteq$

{m | m respects some invariant of N}

This set overapproximates the reachable markings

Place invariants are thus useful to finitely approximate the set of reachable markings

# Place invariant and boundedness

- Theorem: If there exists a place invariant i and a place p s.t. i(p)>0 then p is bounded.
- Remark: the reverse is not true.
  - One can find a bounded net that doesn't have a place invariant i with i(p)>0 for each place.

# Place invariant

- Question: how do we compute them ?

# Matrix characterisation

- The negative effect (consumption) of all the transitions on all the places can be summarised in one matrix:

$$W^- = \left(\begin{array}{cccc} I_1(p_1) & I_2(p_1) & \cdots & I_k(p_1) \\ I_1(p_2) & I_2(p_2) & \cdots & I_k(p_2) \\ \vdots & \vdots & \ddots & \vdots \\ I_1(p_n) & I_2(p_n) & \cdots & I_k(p_n) \end{array}\right) \begin{array}{l} \text{neg. eff. on } p_1 \\ \text{neg. eff. on } p_2 \\ \\ \\ \end{array}$$

where, for any i: $t_i = \langle I_i, O_i \rangle$

# Matrix characterisation

- The same can be done with the positive effects:

$$W^+ = \left(\begin{array}{cccc} O_1(p_1) & O_2(p_1) & \cdots & O_k(p_1) \\ O_1(p_2) & O_2(p_2) & \cdots & O_k(p_2) \\ \vdots & \vdots & \ddots & \vdots \\ O_1(p_n) & O_2(p_n) & \cdots & O_k(p_n) \end{array}\right) \begin{array}{l} \text{pos. eff. on } p_1 \\ \text{pos. eff. on } p_2 \\ \phantom{x} \\ \phantom{x} \end{array}$$

where, for any i: $t_i = \langle I_i, O_i \rangle$

# Incidence Matrix

- The global effect of every transition can be summarised as a single matrix:

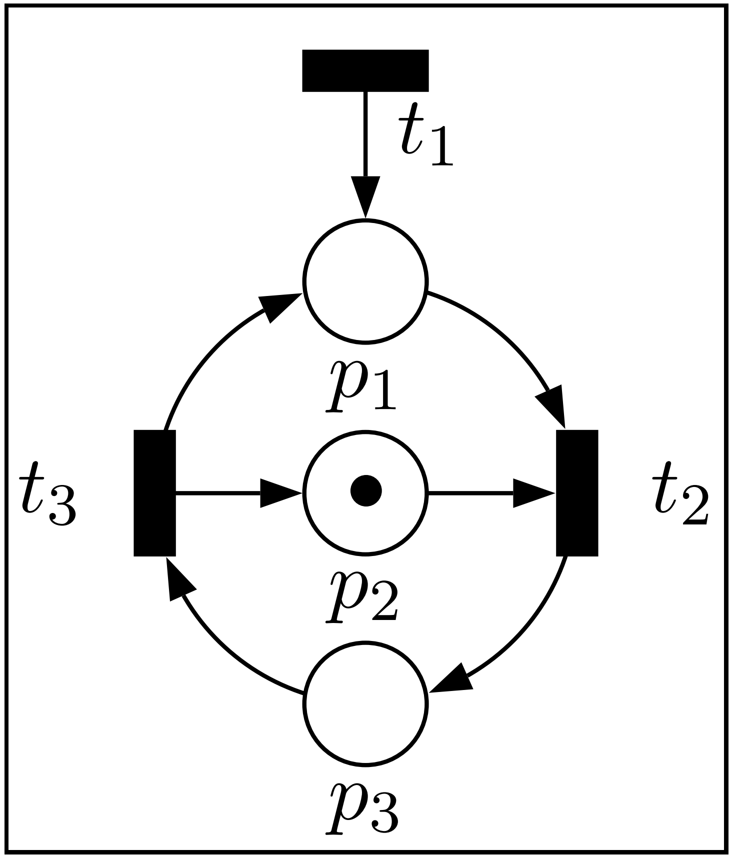
$$W = W^+ - W^-$$

W is called the incidence matrix of the net

# Example

$$W^+ = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad W^- = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

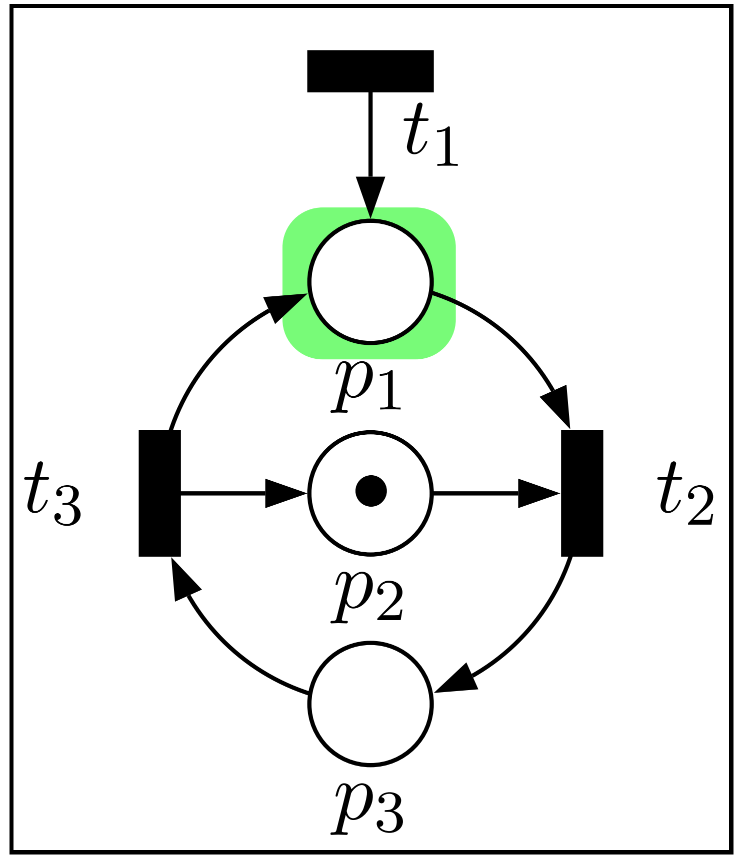
$$W = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$



# Example

$$W^+ = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad W^- = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$W = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$



# Example

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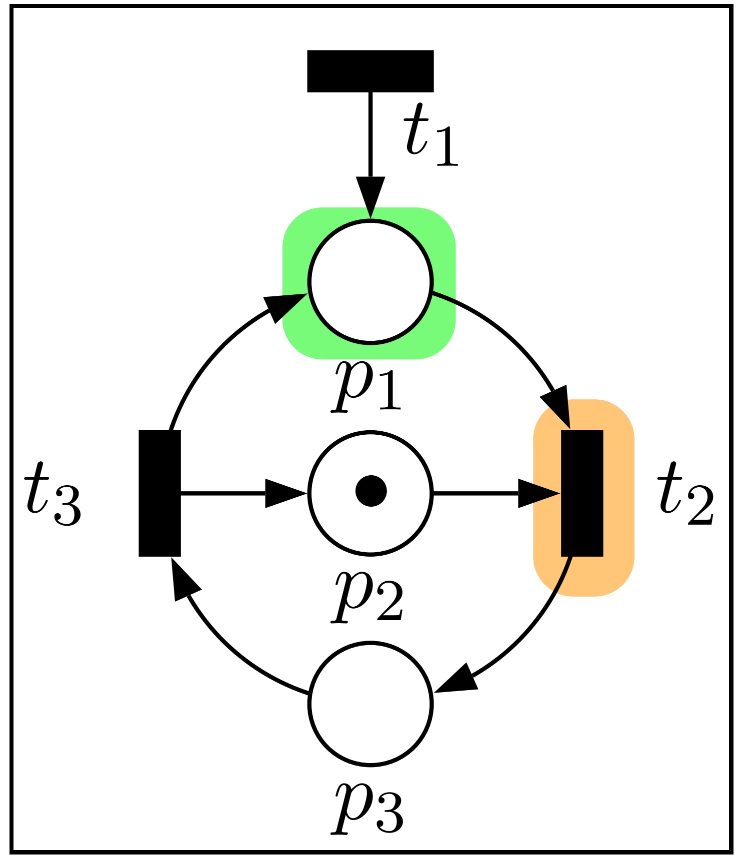

# Computing place invariants

- Intuitively, if i is a place invariant it should assign weights to the places such that the positive and negative effects of every transition are balanced
- Thus, for any transition t = $\langle I, O \rangle$ we should have:

$$\sum_{p \in P} I(p) \times i(p) = \sum_{p \in P} O(p) \times i(p)$$

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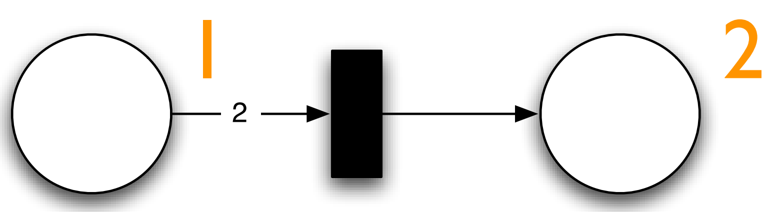

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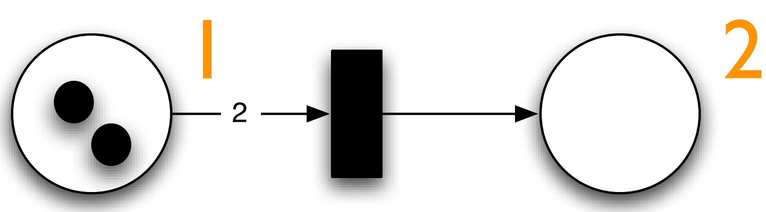

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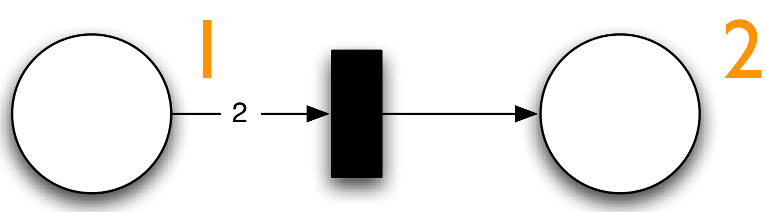

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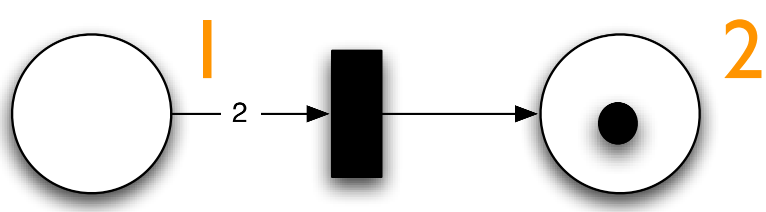

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$$W = \begin{pmatrix} \cdots & O(p_1) - I(p_1) & \cdots \\ \cdots & O(p_2) - I(p_2) & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & O(p_n) - I(p_n) & \cdots \end{pmatrix}$$

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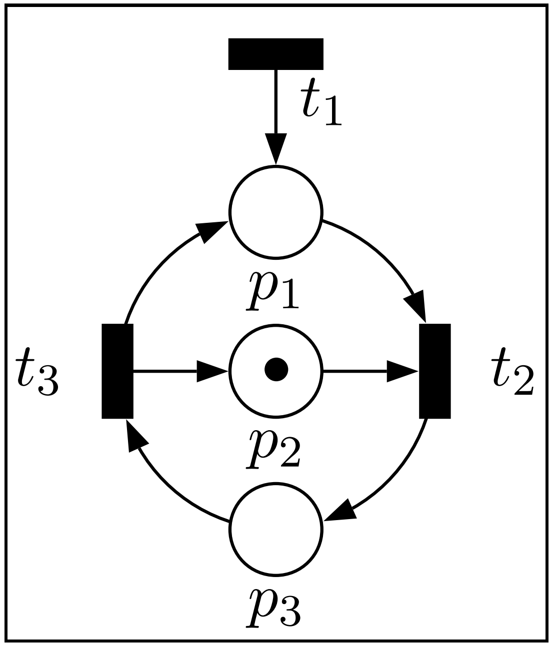
$$i \times W = 0$$

# Example

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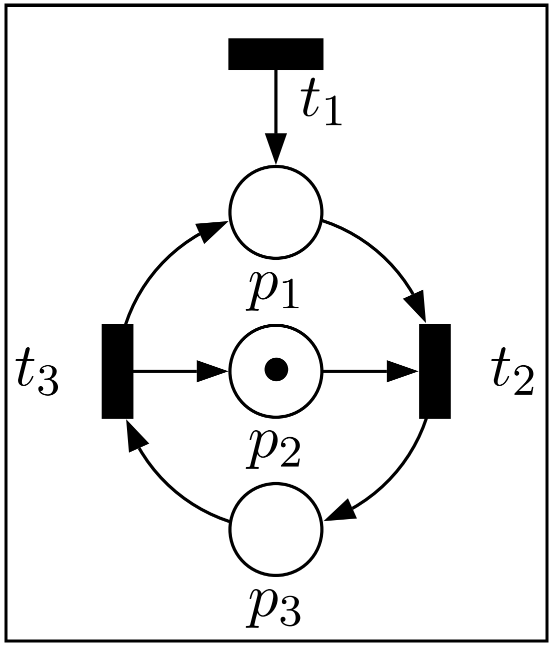


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$$\langle i_1, i_2, i_3 \rangle \times W = 0$$

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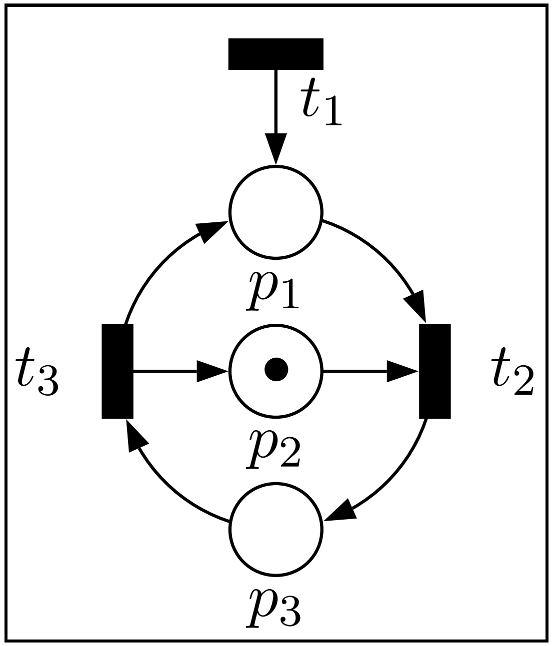


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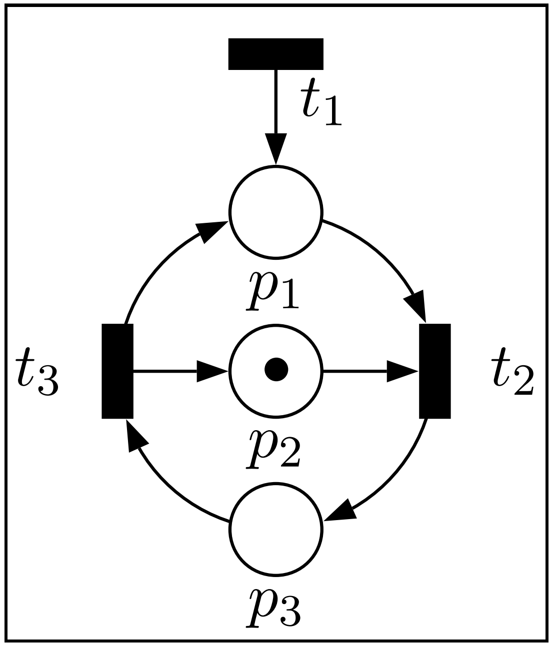


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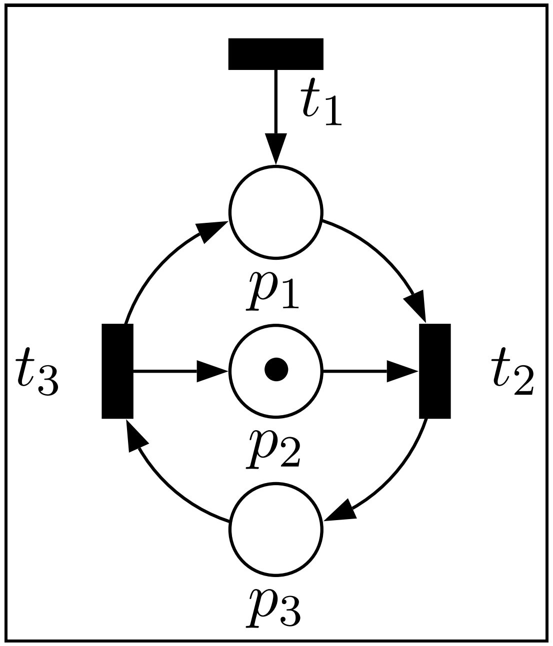


$$W = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & 1 \\ & & -1 \end{pmatrix}$$

Any vector of the form

⟨0, i, i⟩

is a place invariant

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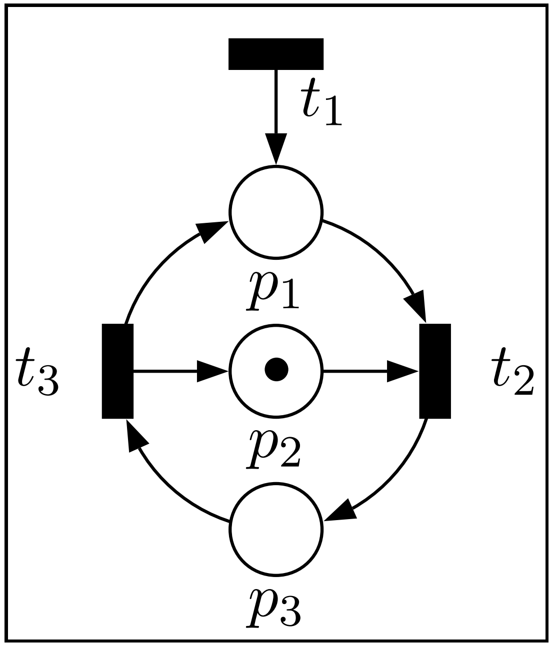
# Proving properties




Let us choose $\langle 0, 1, 1 \rangle$
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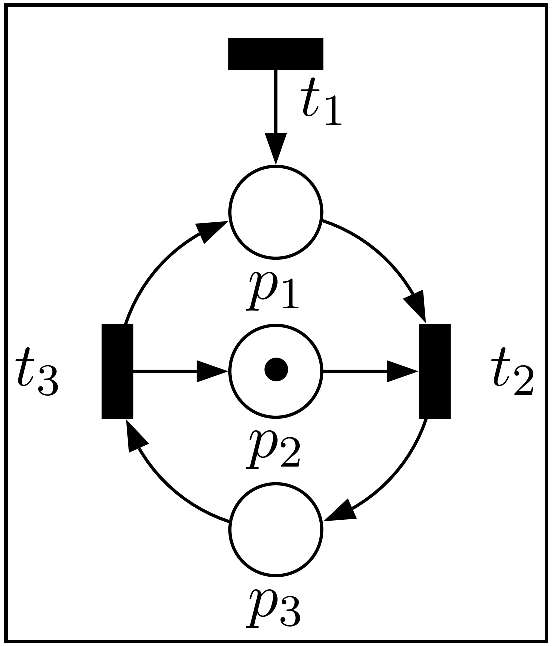


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This means that $p_2$ and $p_3$ are bounded !

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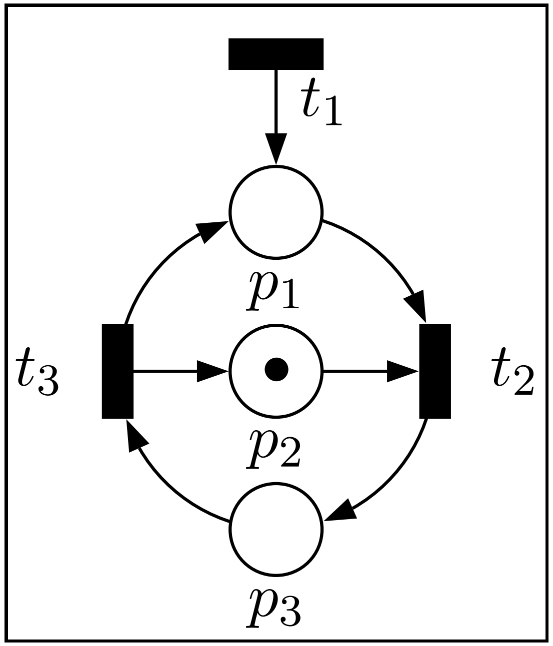


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For any reachable marking m:

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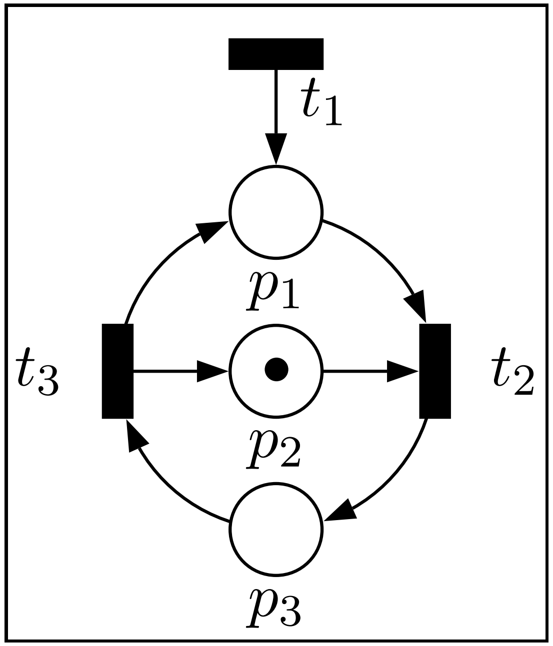


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Hence, mutual exclusion is enforced !

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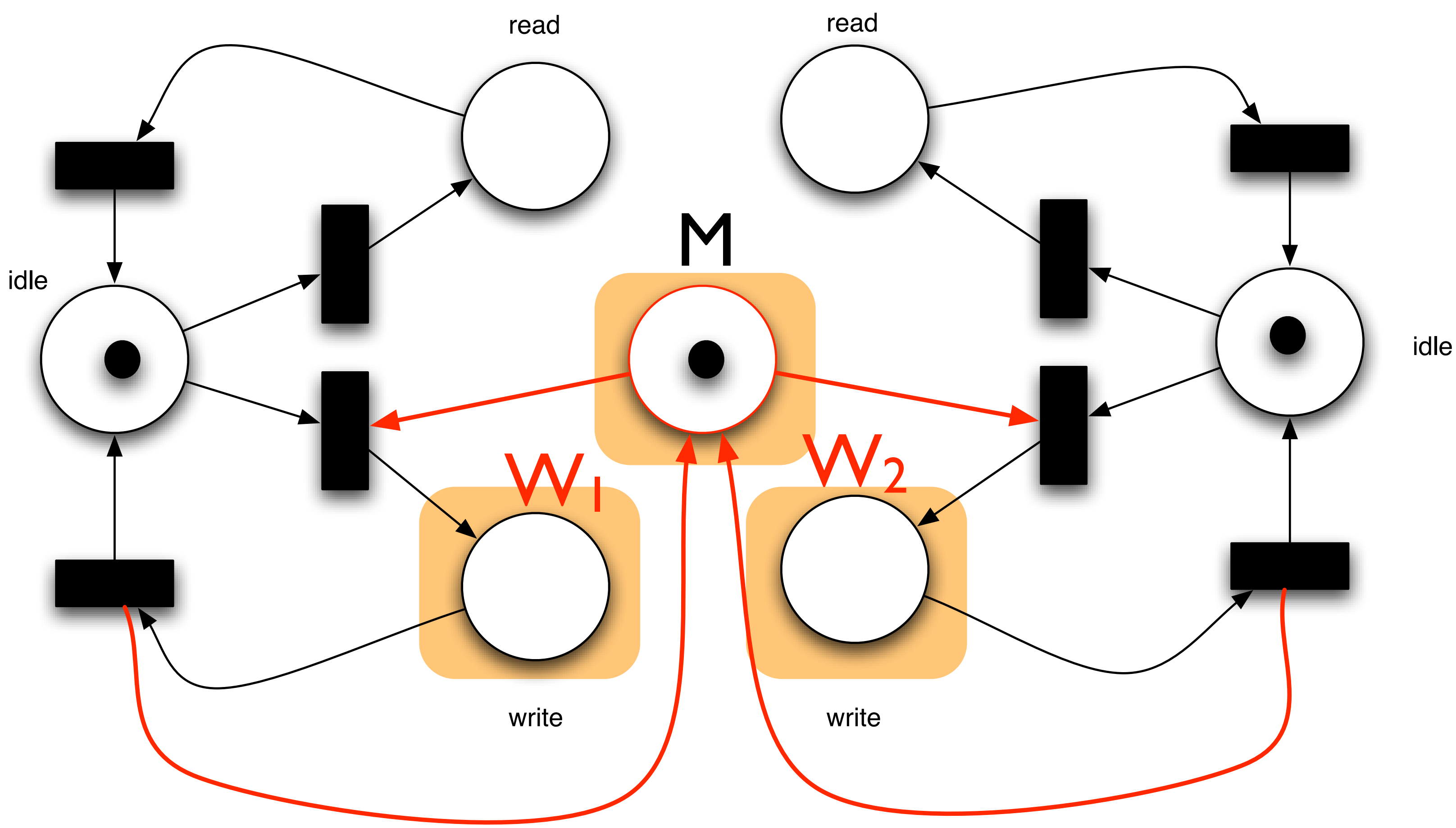


$i(M) = i(W_1) = i(W_2) = 1$ and $i(p) = 0$ otherwise
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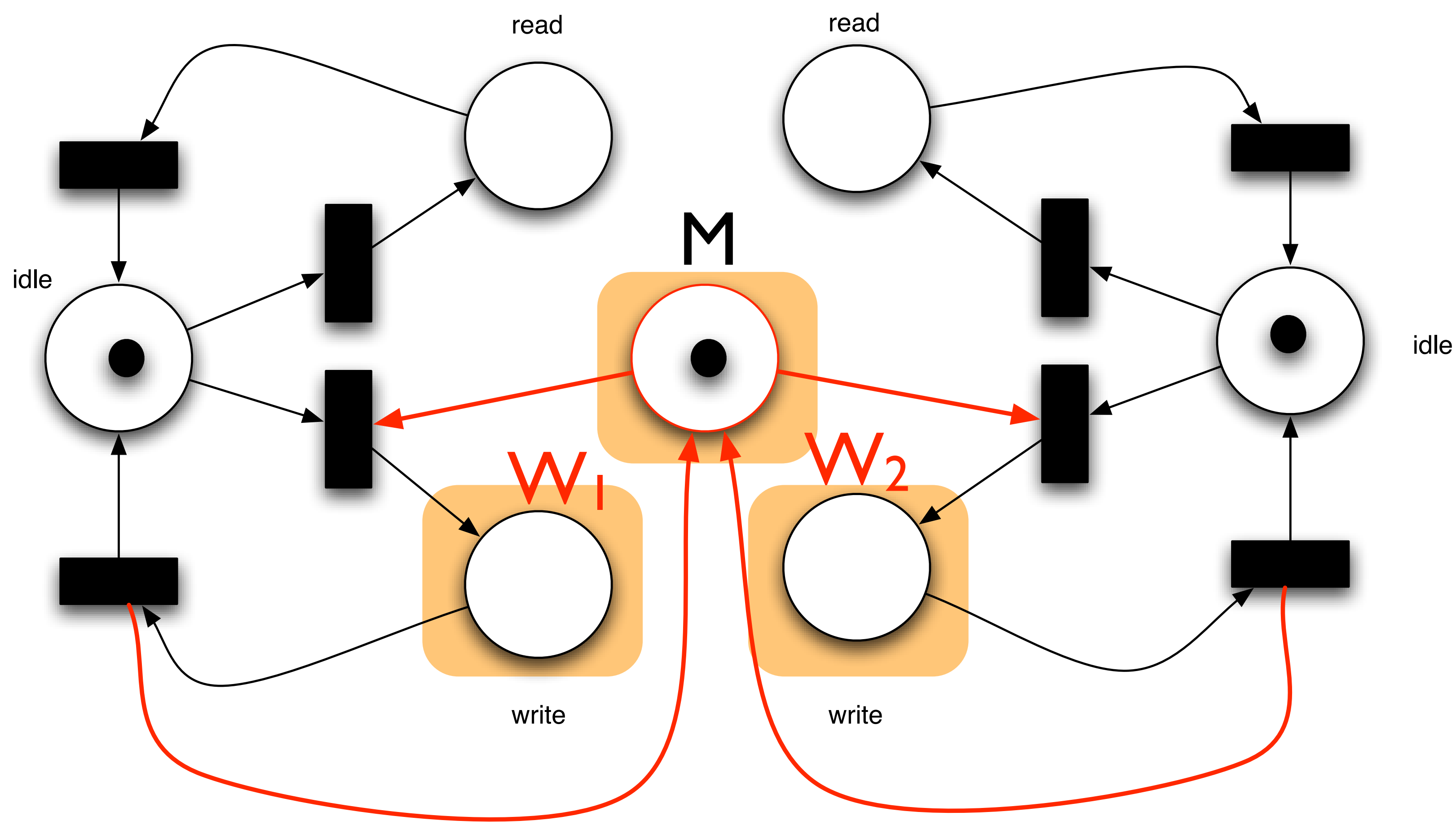


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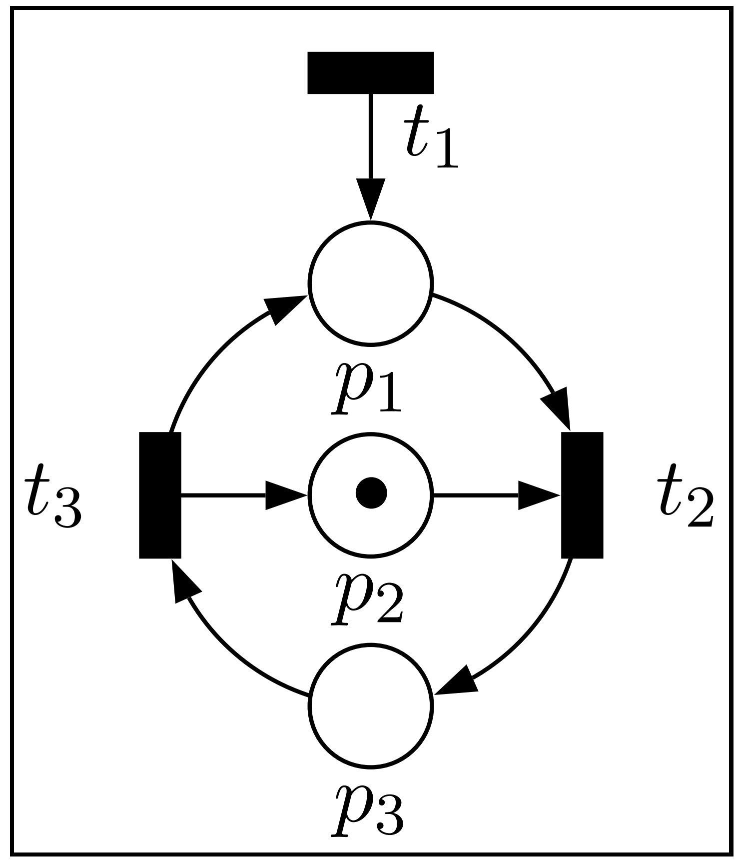
# Karp & Miller
# and
# the coverability set

# The reachability tree revisited

- Reminder: reachability trees can be infinite

$$\begin{tikzcd}
\langle 0p_1,p_2\rangle \arrow[d] & \\
\langle 1p_1,p_2\rangle \arrow[d] \arrow[dr] & \\
\langle 2p_1,p_2\rangle \arrow[d] \arrow[dr] & \langle p_3\rangle \\
\langle 3p_1,p_2\rangle \arrow[d, dotted] & \langle p_1,p_3\rangle \arrow[d, dotted] \\
{} & {}
\end{tikzcd}$$



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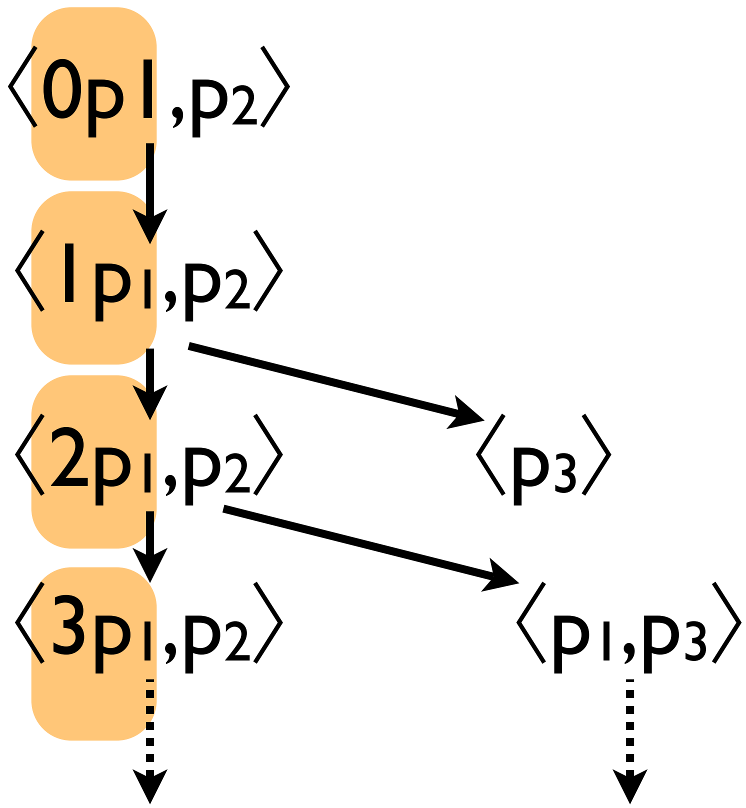


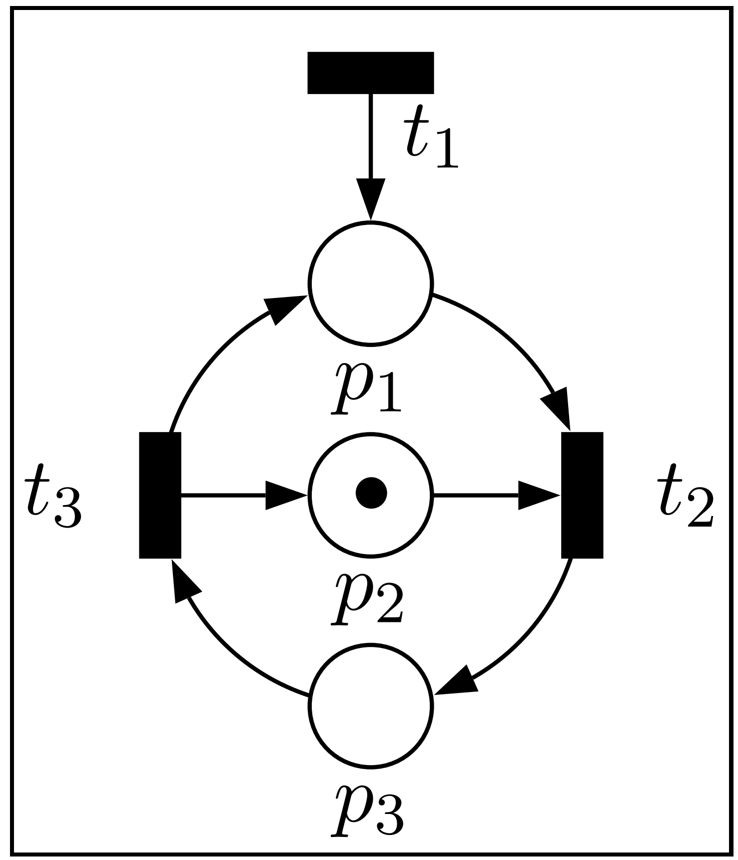

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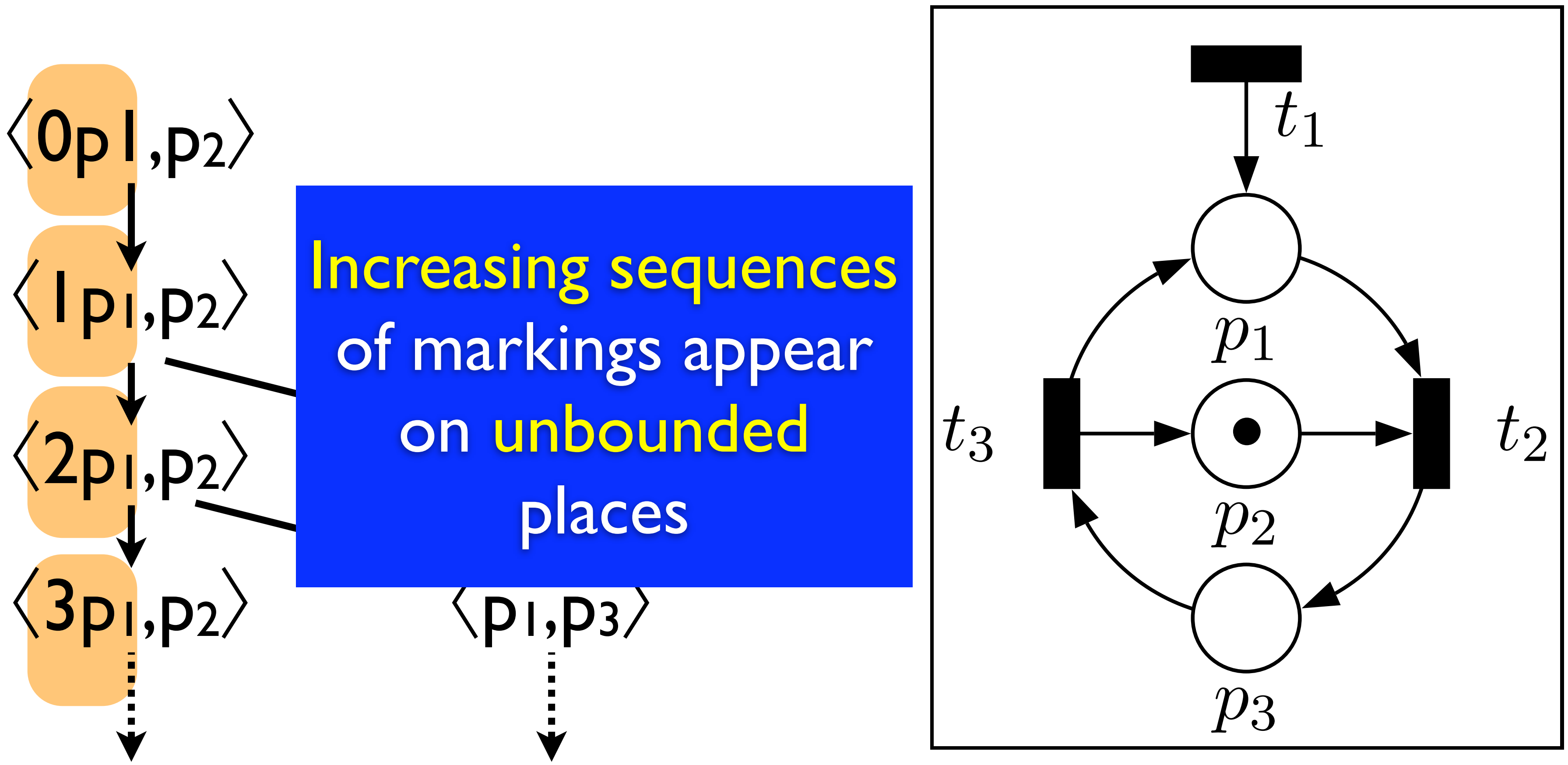

# The reachability tree revisited

- Let us summarise this infinite sequence

$$\langle 0p_1, p_2 \rangle$$
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$$\vdots$$

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\end{array}
\quad \xrightarrow{\quad \text{limit} \quad}
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\quad \xrightarrow{\text{limit}} \quad \langle \omega p_1, p_2 \rangle
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- Let us summarise this infinite sequence

$\omega$ must be regarded as: "any number of tokens"

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# The reachability tree revisited

- Let us summarise this infinite sequence

$$\langle 0p_1, p_2\rangle \to \langle 1p_1, p_2\rangle \to \langle 2p_1, p_2\rangle \to \langle 3p_1, p_2\rangle \dashrightarrow$$

$$\xrightarrow{\text{limit}} \langle \omega p_1, p_2\rangle$$

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Main idea of the Karp and Miller algorithm

# Karp & Miller

- Propose in 1969 a solution to detect unbounded places of a Petri net

# Monotonicity

- Petri nets induce (strongly) monotonic transition systems:

$$\begin{tikzcd} m_3 \arrow[d, phantom, "\curlyvee"] & \\ m_1 \arrow[r, "t"] & m_2 \end{tikzcd}$$

- In particular:

if

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- Petri nets induce (strongly) monotonic transition systems:

$$\begin{array}{ccc} m_3 & & m_4 \\ \succcurlyeq & & \\ m_1 & \xrightarrow{t} & m_2 \end{array}$$

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- In particular:

if $\langle i_1, i_2, i_3 \rangle$ $\langle i'_1, i'_2, i'_3 \rangle$

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$$
\begin{tikzcd}
m_3 \arrow[r, "t"] \arrow[d, phantom, "\rotatebox[origin=c]{-90}{$\succ$}"] & m_4 \arrow[d, phantom, "\rotatebox[origin=c]{-90}{$\succ$}"] \\
m_1 \arrow[r, "t"] & m_2
\end{tikzcd}
$$

- In particular:

if $\langle i_1, i_2, i_3 \rangle \longrightarrow \langle i_1', i_2', i_3' \rangle$

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- In particular:

if $$\begin{tikzcd} \langle i_1, i_2, i_3 \rangle \arrow[r, "\prec"] \arrow[r, bend right, dash, "<"'] & \langle i'_1, i'_2, i'_3 \rangle \end{tikzcd}$$

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- In particular:

if $\langle i_1, i_2, i_3 \rangle \xrightarrow{\ \prec\ } \langle i'_1, i'_2, i'_3 \rangle$

$<$

then $p_2$ is unbounded

# Example

$\langle 1, 0, 0, 0 \rangle$

# Example

$$
\begin{tikzcd}
\langle 1, 0, 0, 0 \rangle \arrow[dr, dotted] & \\
& \langle 0, 0, 1, 0 \rangle
\end{tikzcd}
$$

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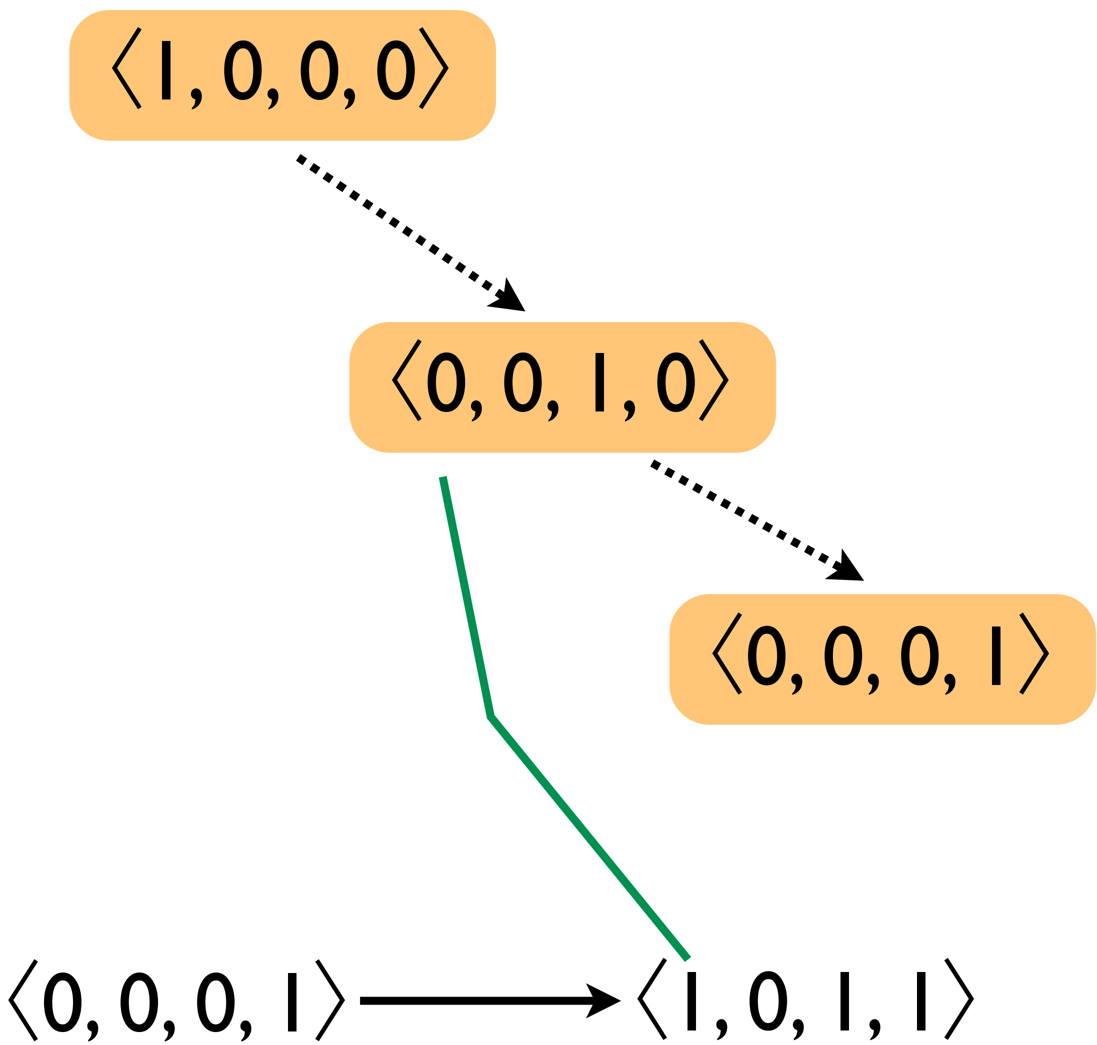
$$\dashrightarrow \langle 0, 0, 1, 0 \rangle$$

$$\dashrightarrow \langle 0, 0, 0, 1 \rangle$$

$$\langle 0, 0, 0, 1 \rangle \longrightarrow \langle 1, 0, 1, 1 \rangle$$

# Example



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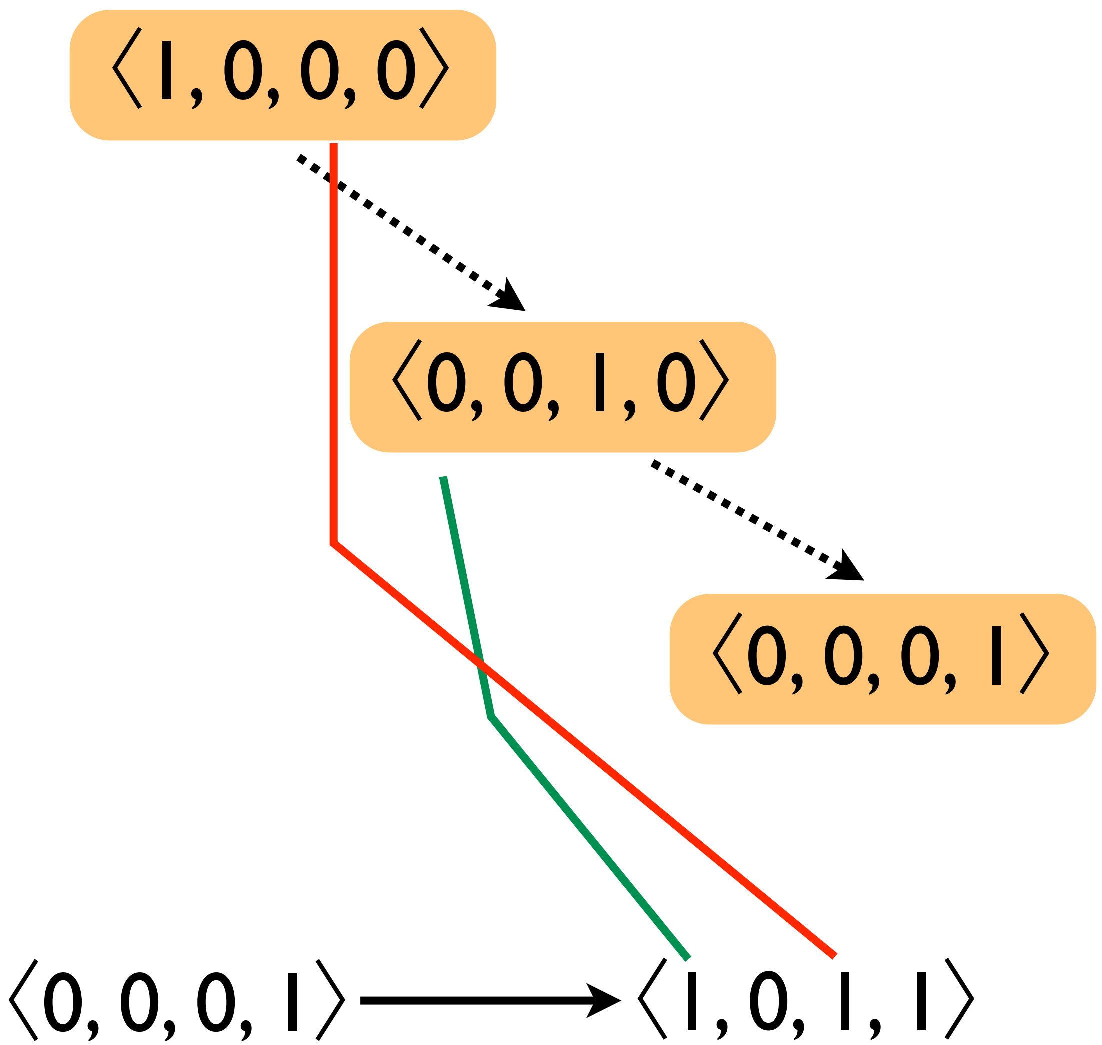

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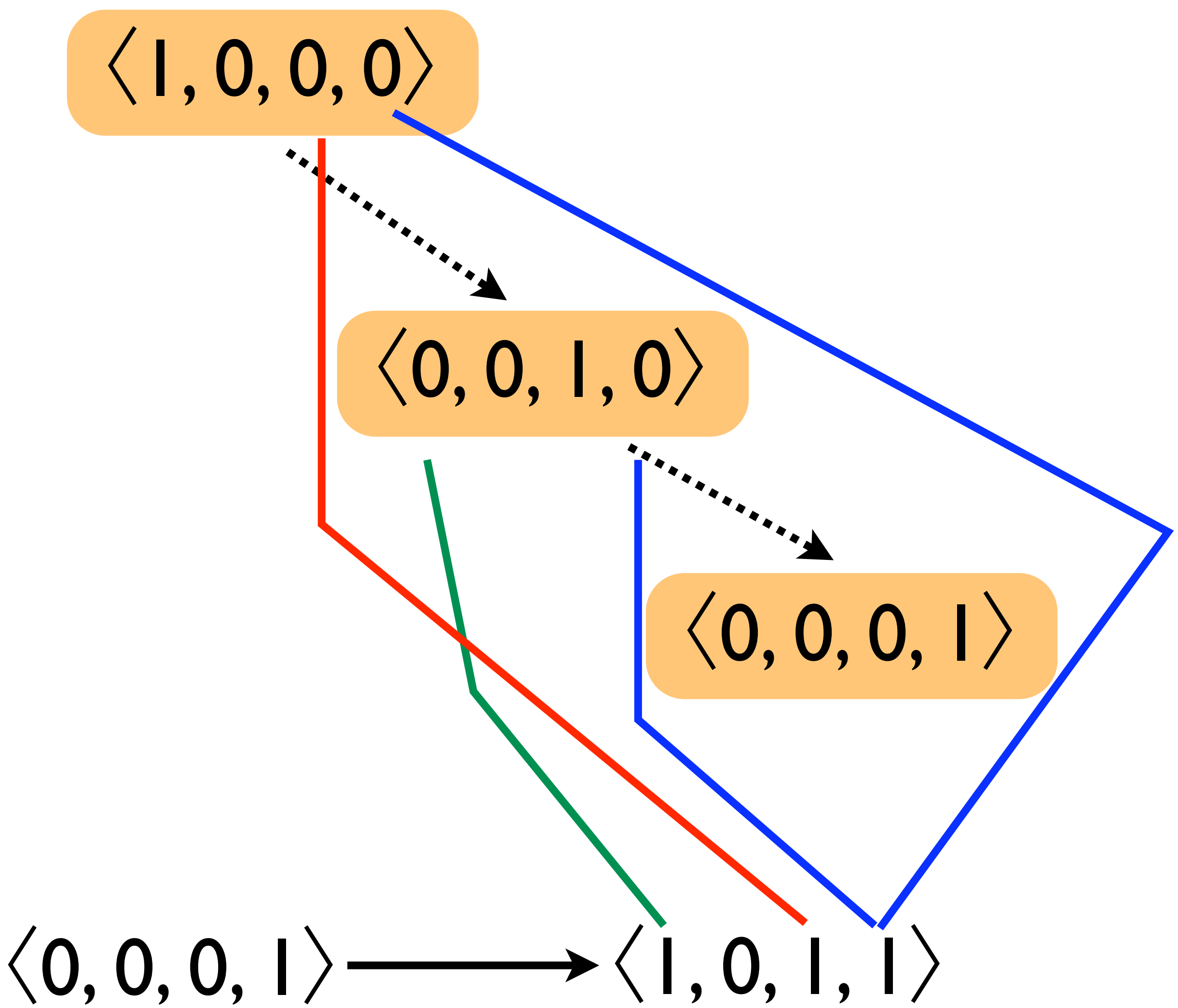

$\langle 1, 0, 0, 0 \rangle$
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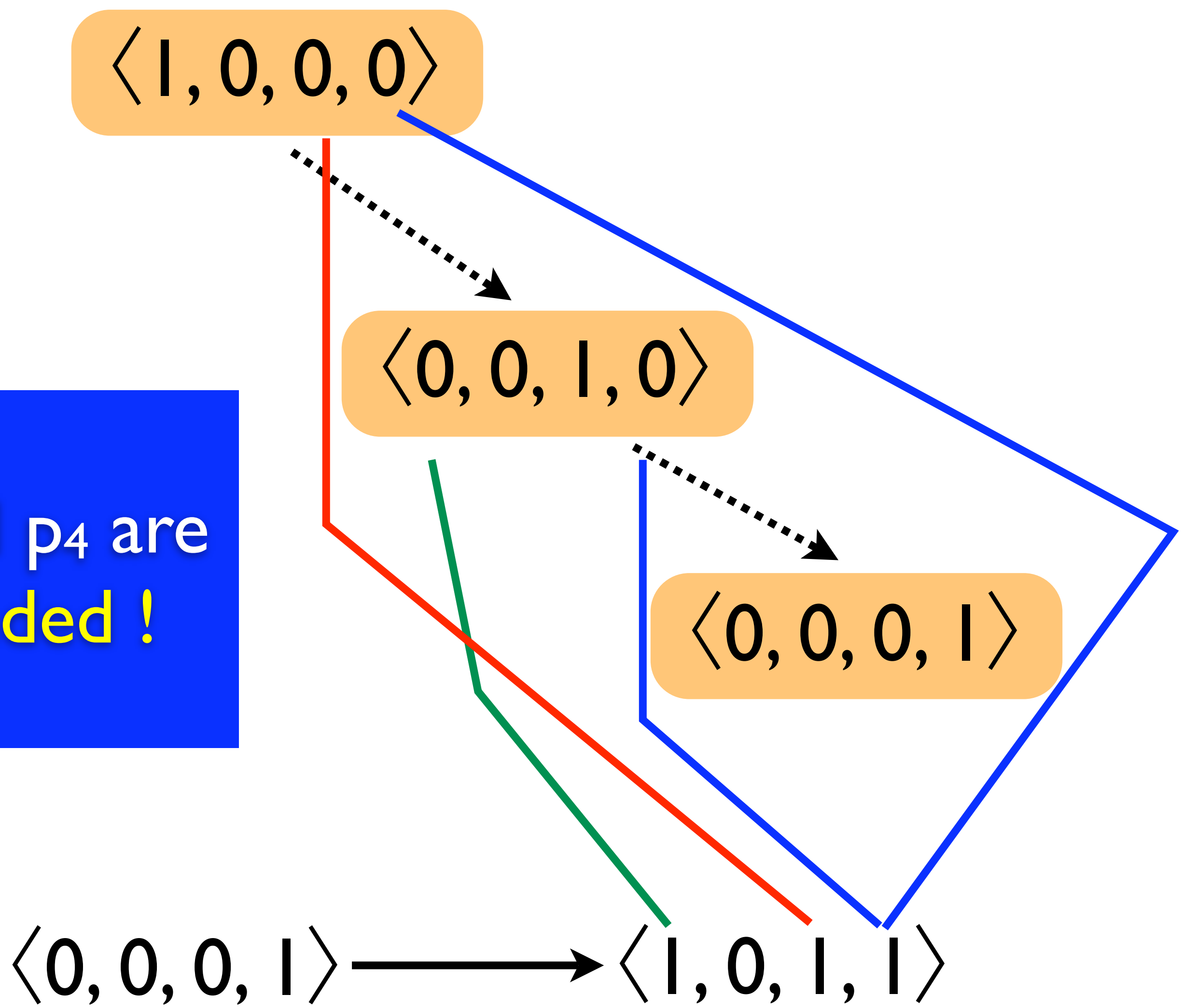


$p_1$, $p_3$ and $p_4$ are unbounded !

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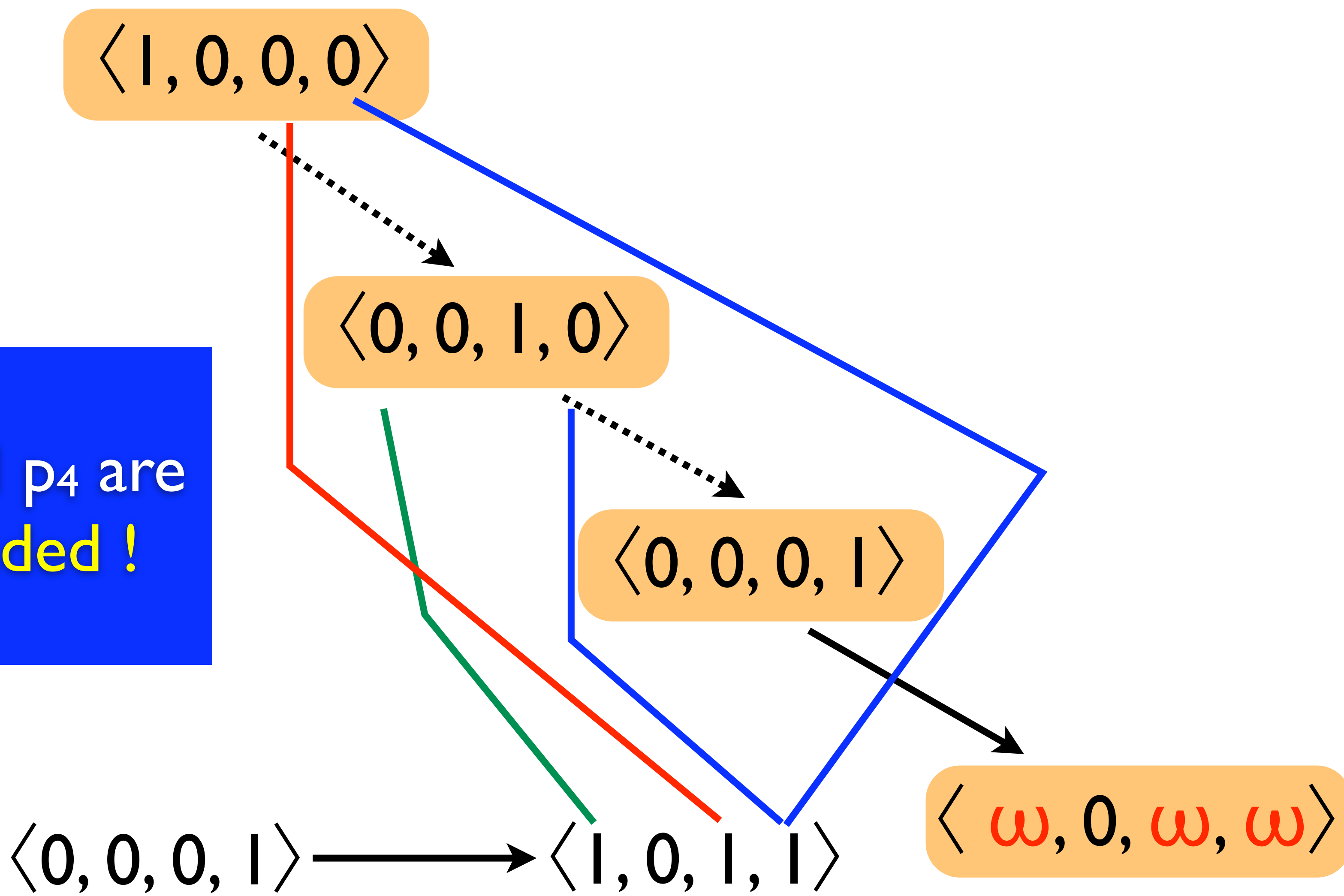


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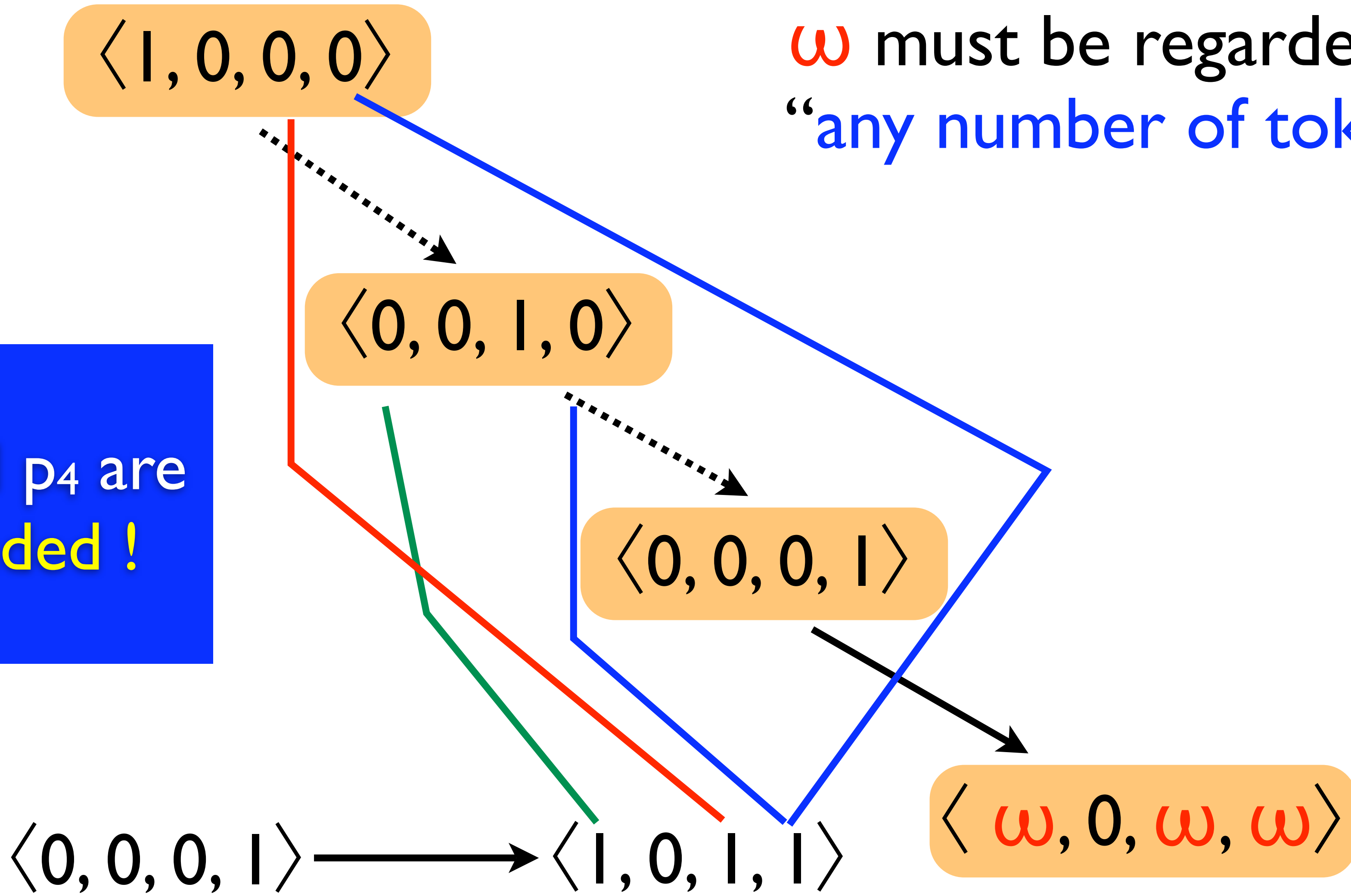


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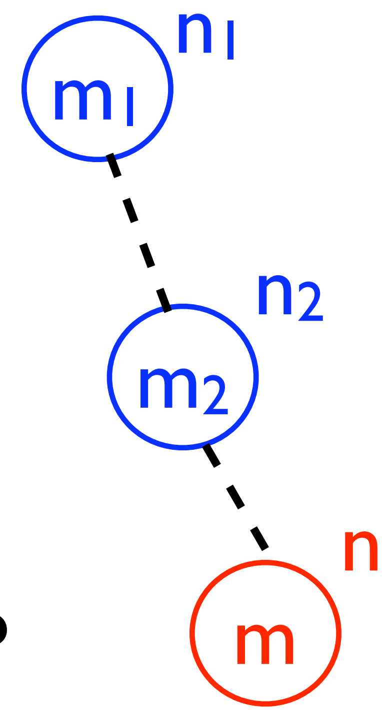
# Karp & Miller Acceleration

This is how we compute the successors of a node n:

**foreach** *Successor* $m'$ *of* $m$ **do**
$\quad m_\omega \leftarrow m'$;
$\quad$**foreach** *ancestor* $n_i$ *s.t.* $m_i \prec m'$ **do**
$\quad\quad$**foreach** *place* $p$ *s.t.* $m_i(p) < m'(p)$ **do**
$\quad\quad\quad m_\omega(p) \leftarrow \omega$;
$\quad$Add $m_\omega$ as child of $n$;



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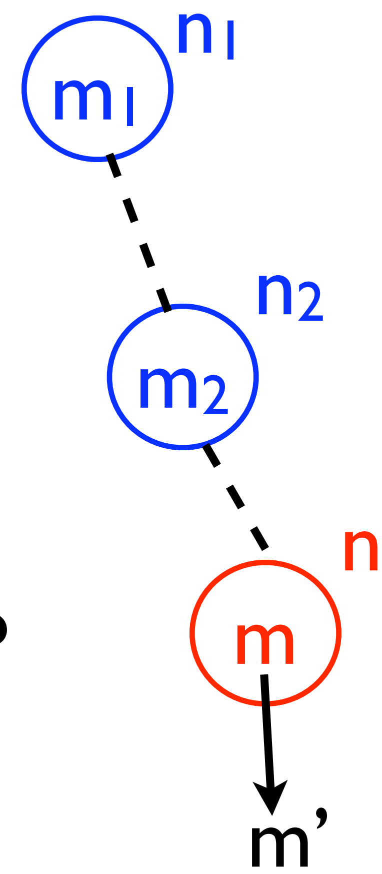

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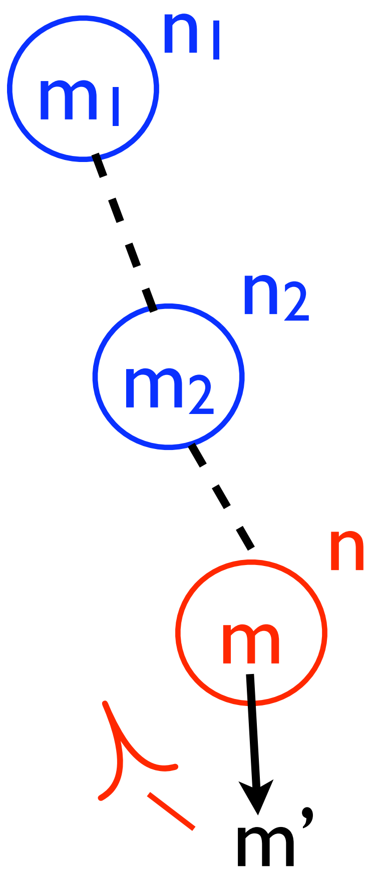

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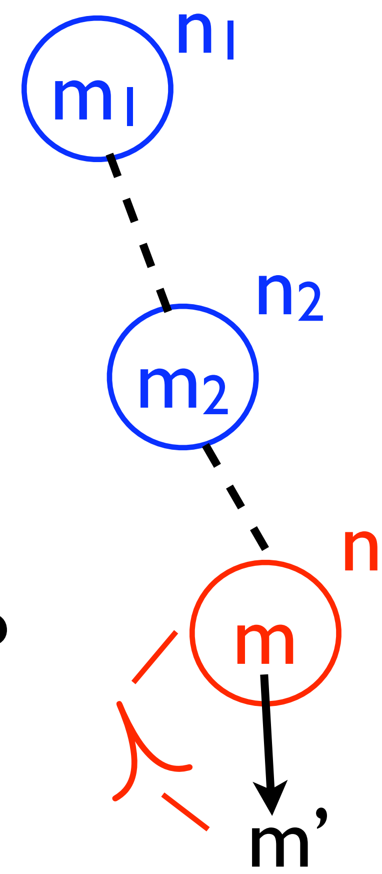

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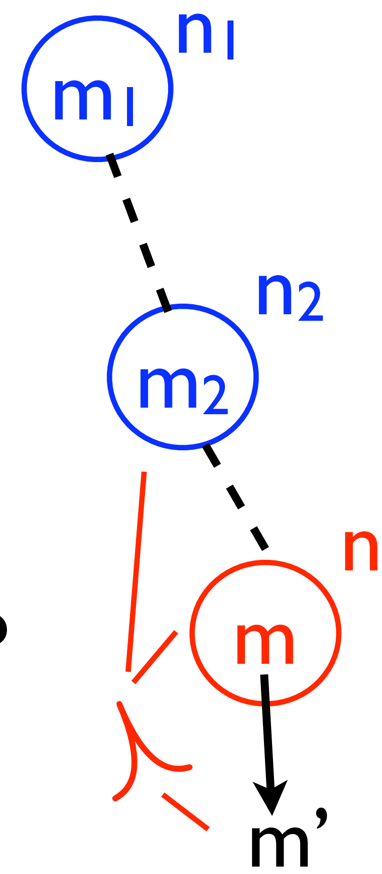

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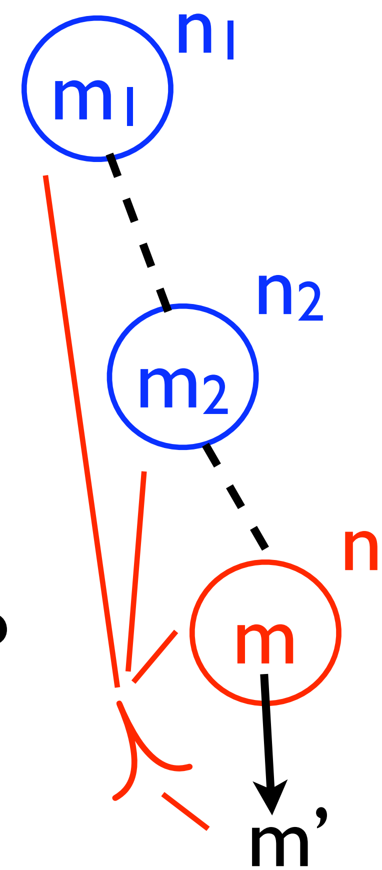

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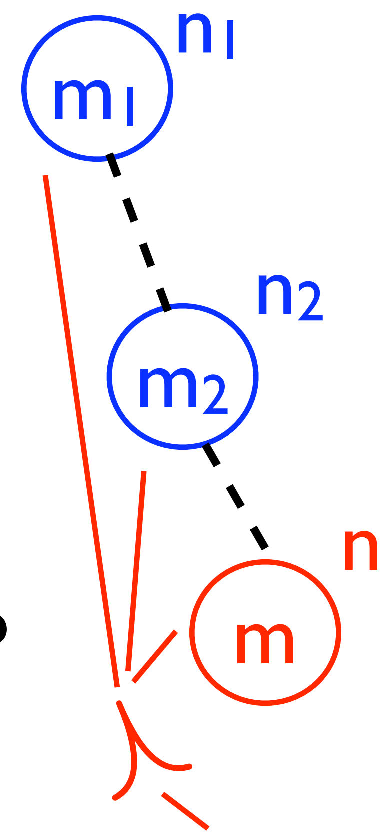

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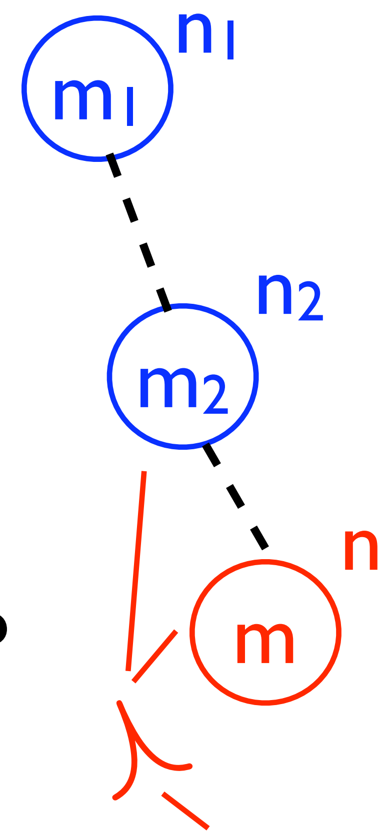

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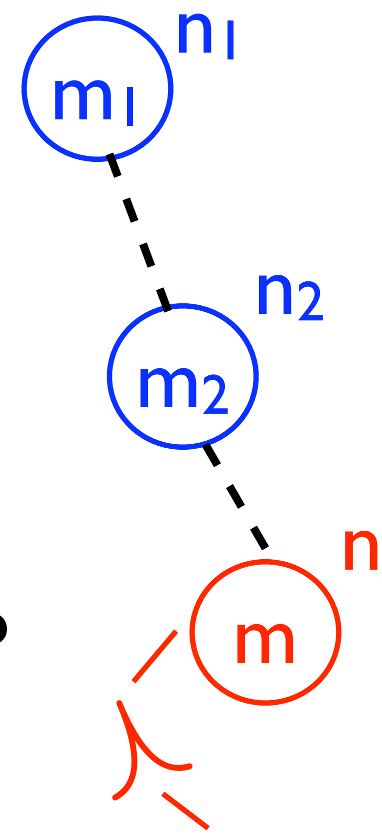

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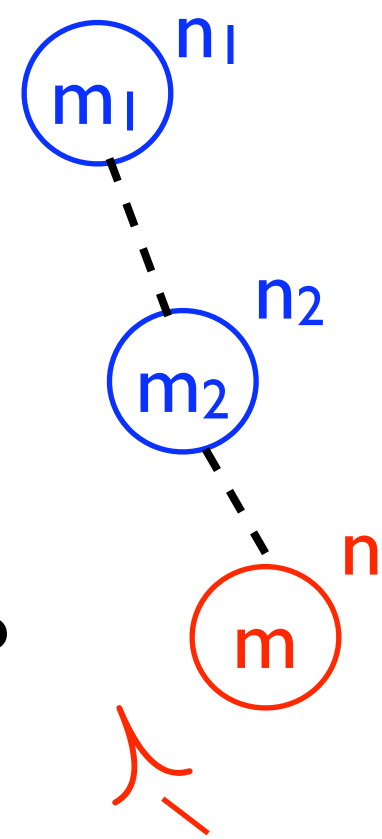

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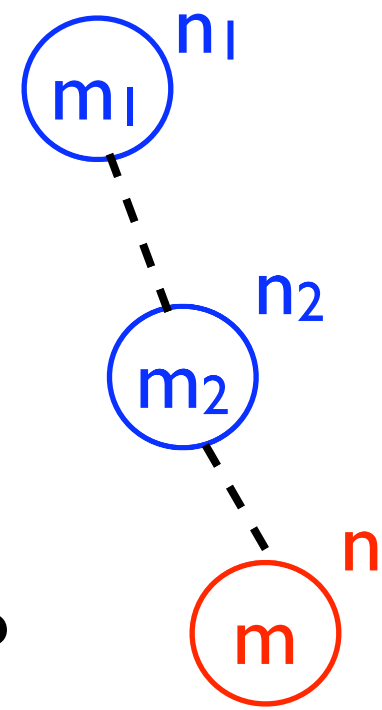

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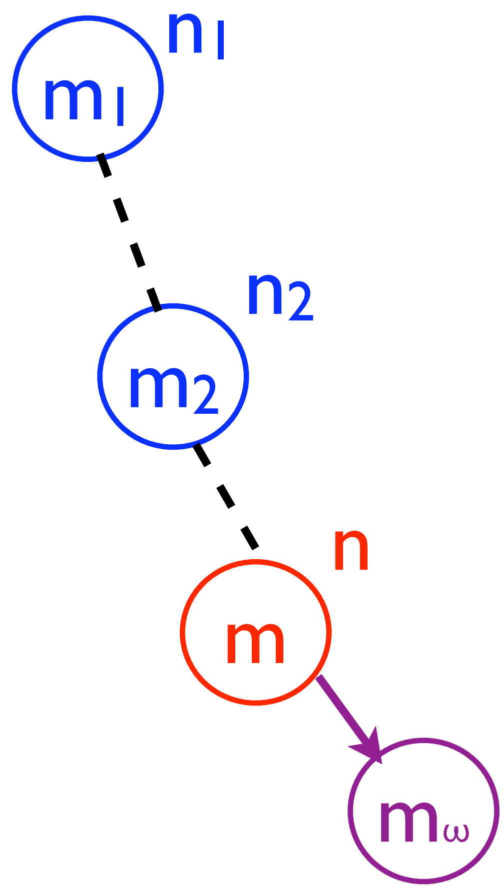
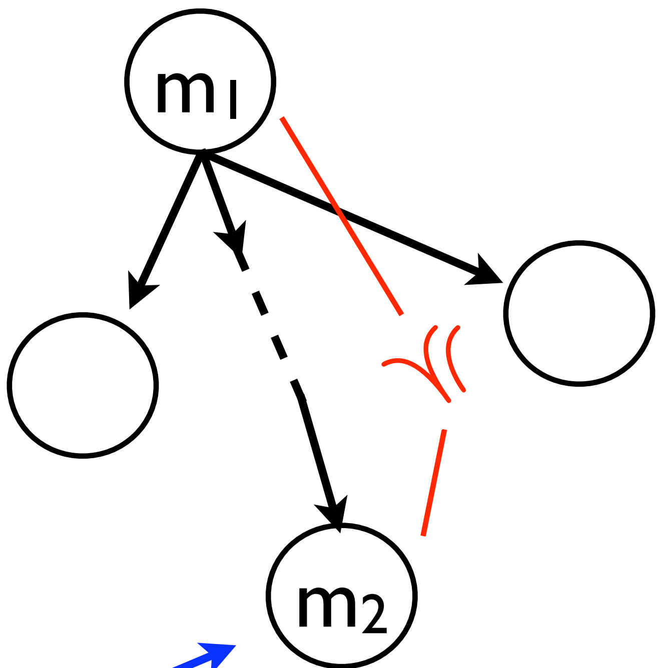


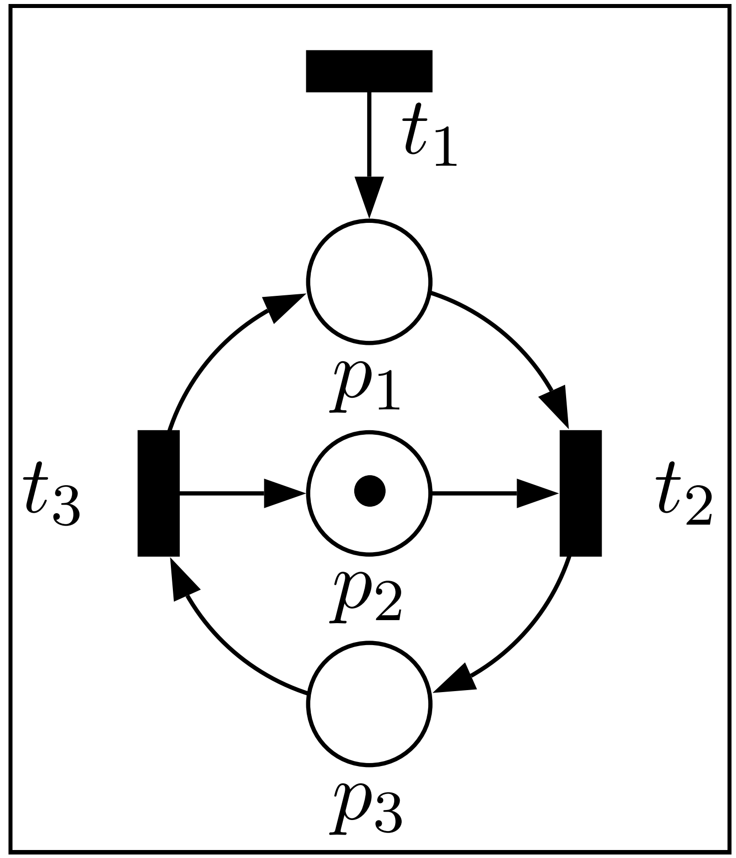
# Karp & Miller
# Stopping a branch




This node doesn't have to be developed

# Example of K&M tree

$\langle 0, 1, 0 \rangle$



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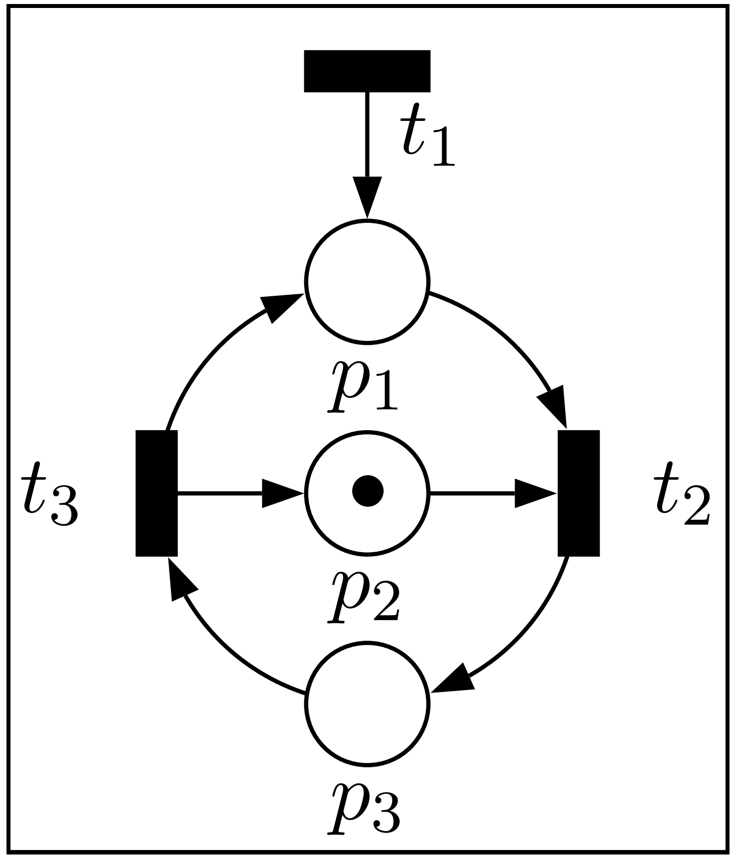
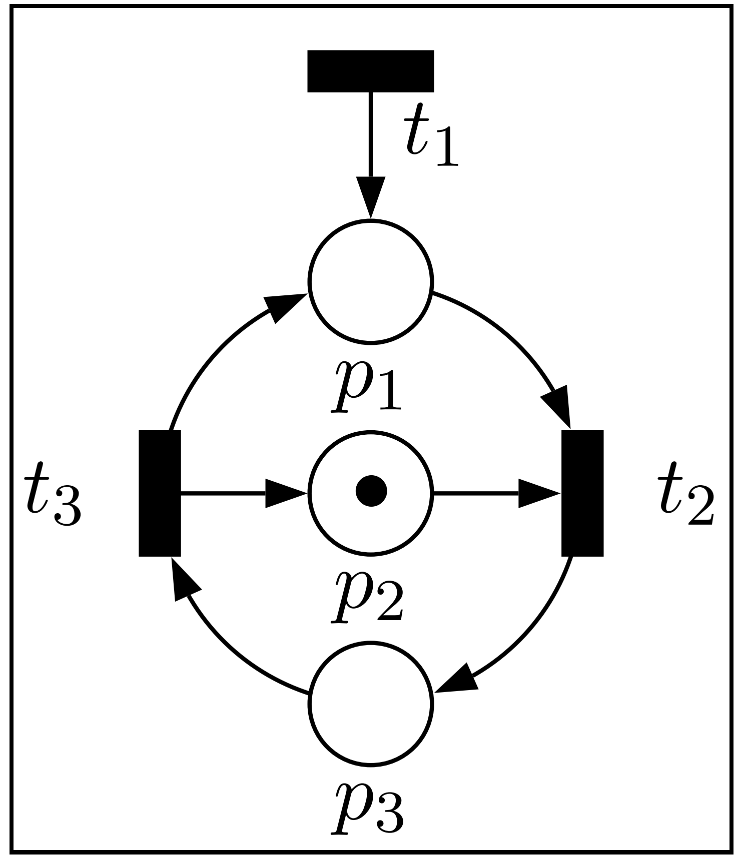



$$(0,1,0) \xrightarrow{t_1} (1,1,0) \succ (0,1,0)$$

# Example of K&M tree

$$\langle 0, 1, 0\rangle$$
$$\Big|\ t_1$$
$$\langle \omega, 1, 0\rangle$$

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# Example of K&M tree

$$\begin{tikzcd}
& \langle 0, 1, 0 \rangle \arrow[d, dash, "t_1"] \\
& \langle \omega, 1, 0 \rangle \arrow[dl, dash, "t_1"'] \\
\langle \omega, 1, 0 \rangle &
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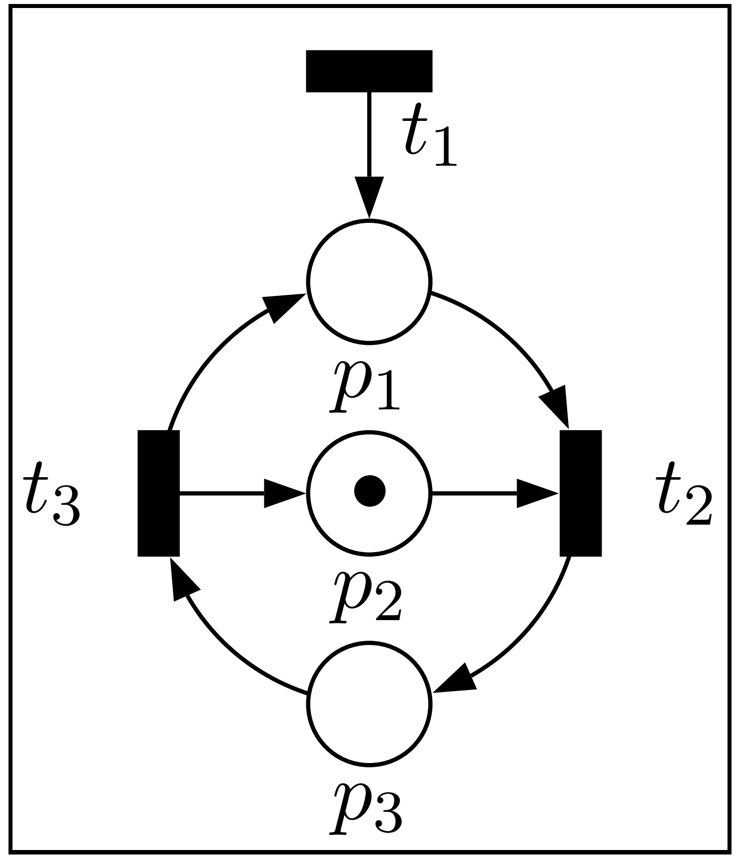


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\langle \omega, 1, 0 \rangle & & \langle \omega, 0, 1 \rangle
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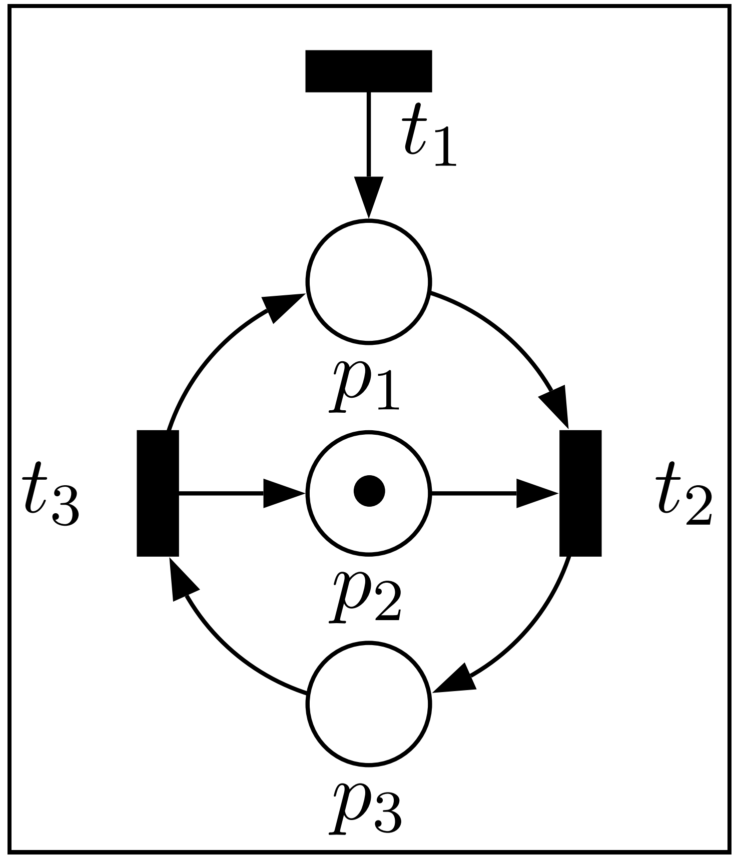
$$

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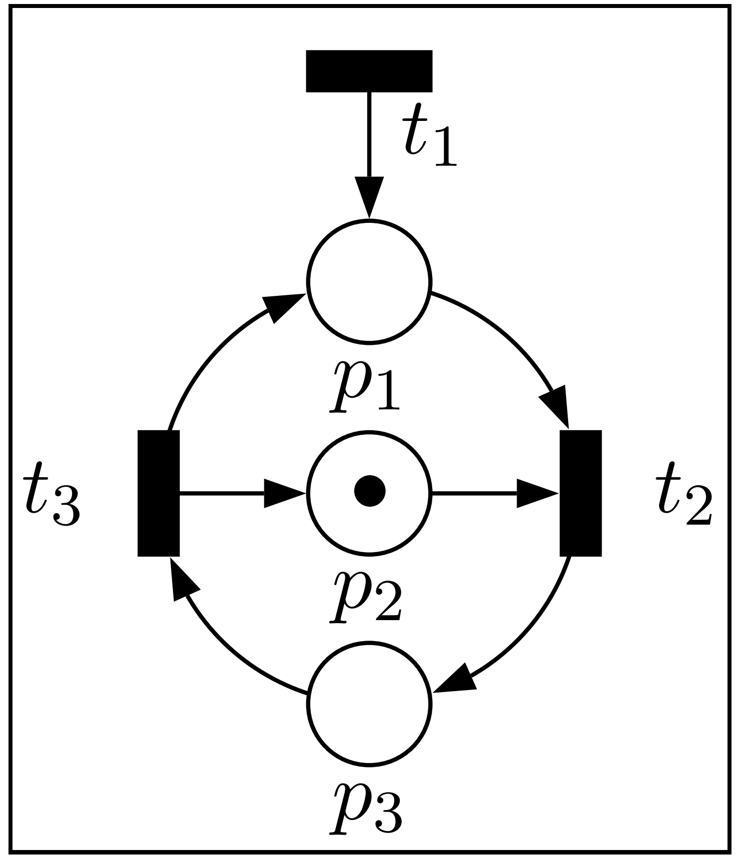


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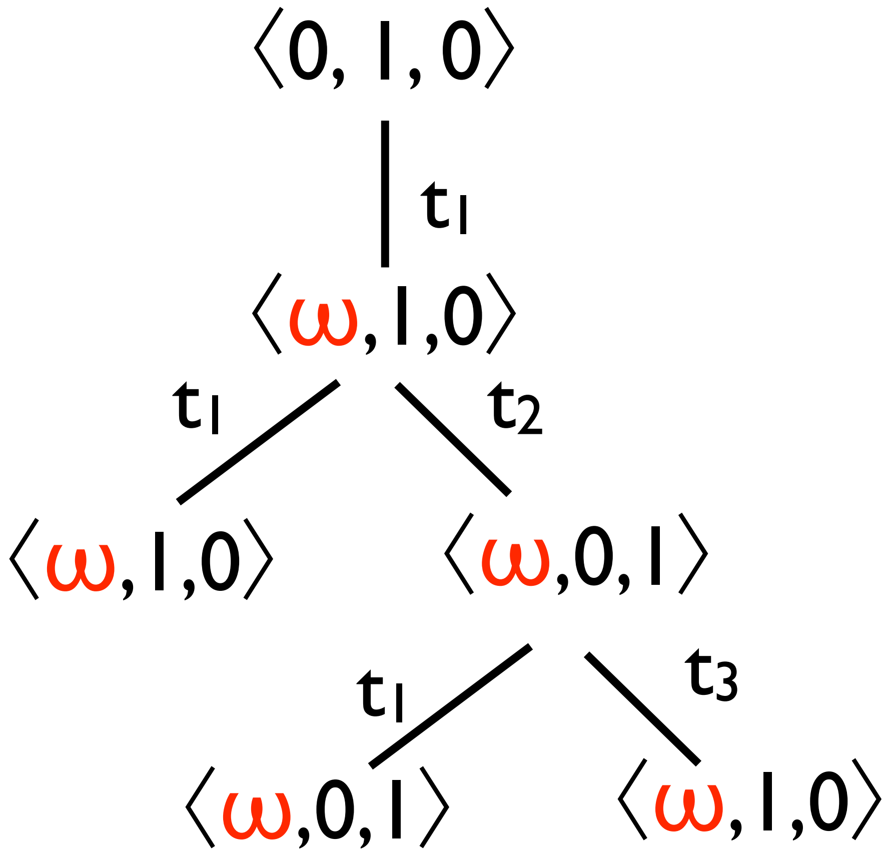
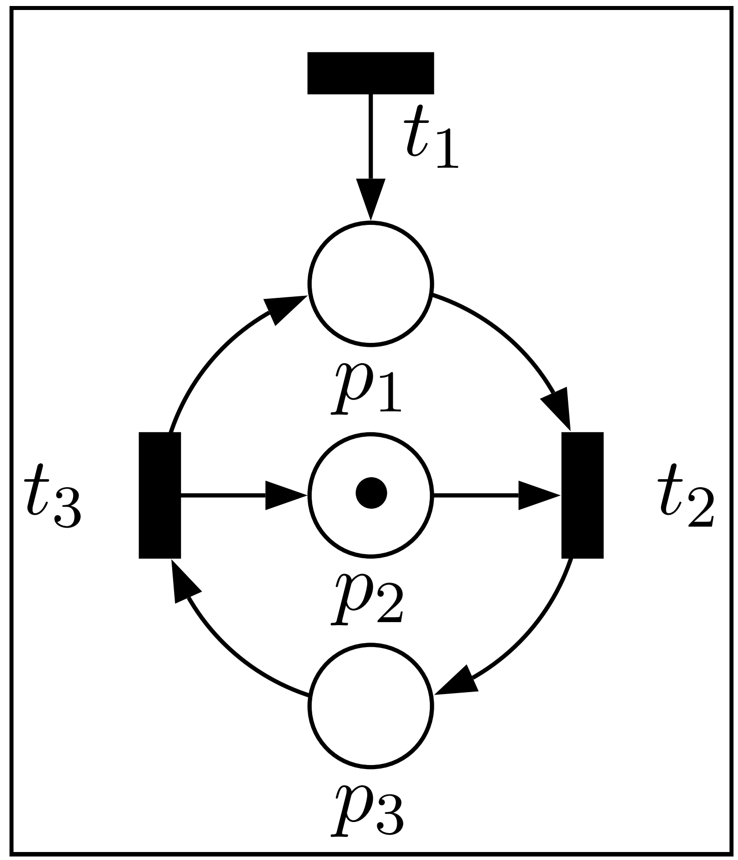



$$(0,1,0) \xrightarrow{t_1} (1,1,0) \succ (0,1,0)$$

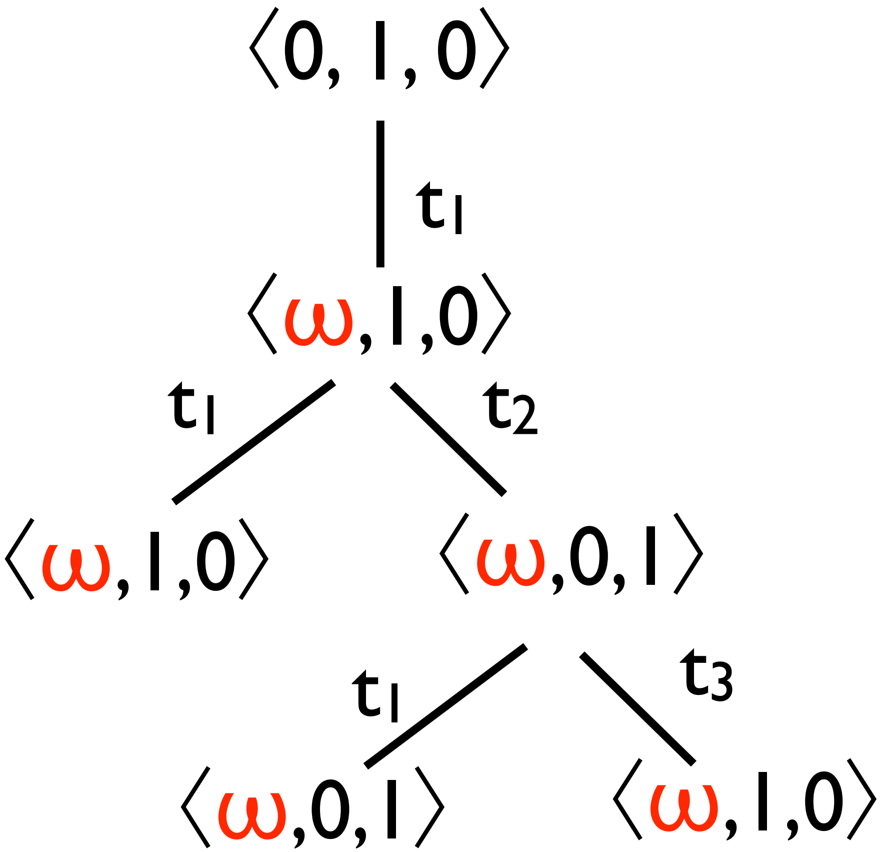
# Properties

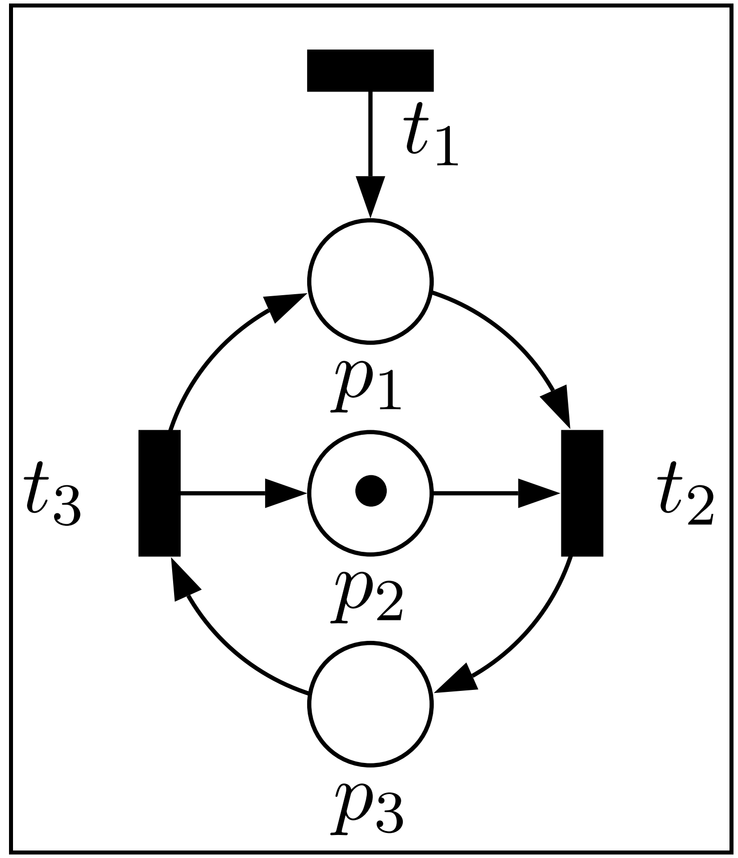
- Theorem: the K&M tree is always finite.
  - Idea of the proof:
    - if the net is not bounded, it is because of some infinite increasing sequence of markings.
    - such sequences are detected in a finite amount of time by adding $\omega$ in the unbounded places.

# Properties

- Theorem: a net is bounded iff there is no node containing an $\omega$ in its K&M tree.
- Theorem: place p is unbounded iff there exists a node labeled by m in the K&M tree s.t. $m(p) = \omega$.
- Theorem: transition t is semi-live iff there exists a node labeled by m in the K&M tree s.t. t can fire in m.

# Example



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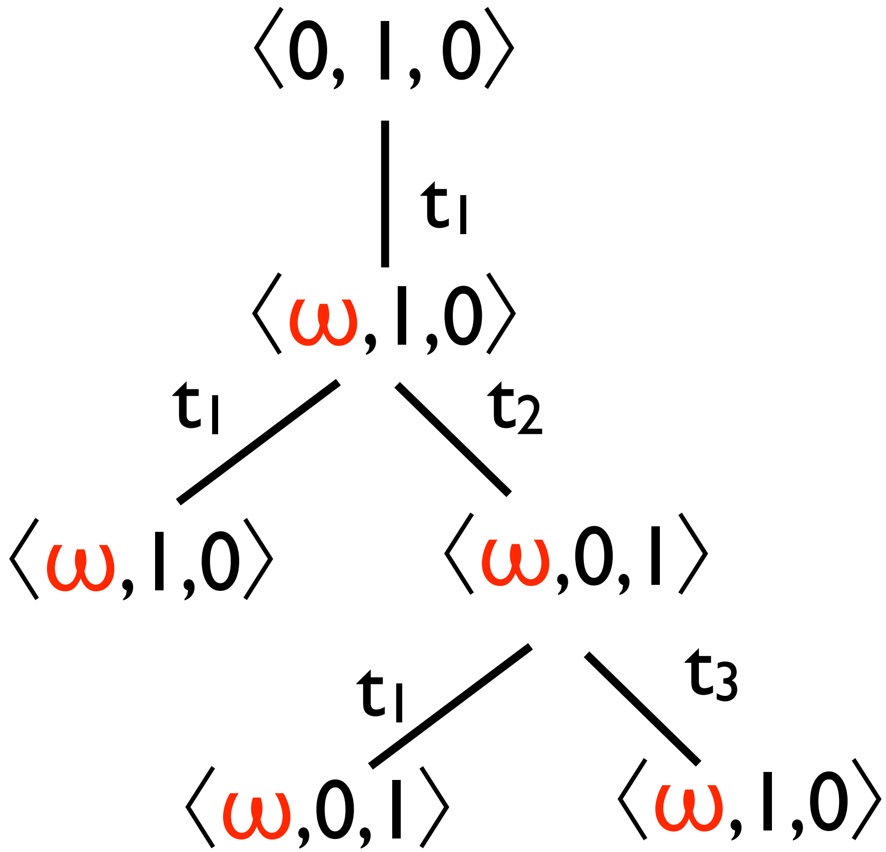
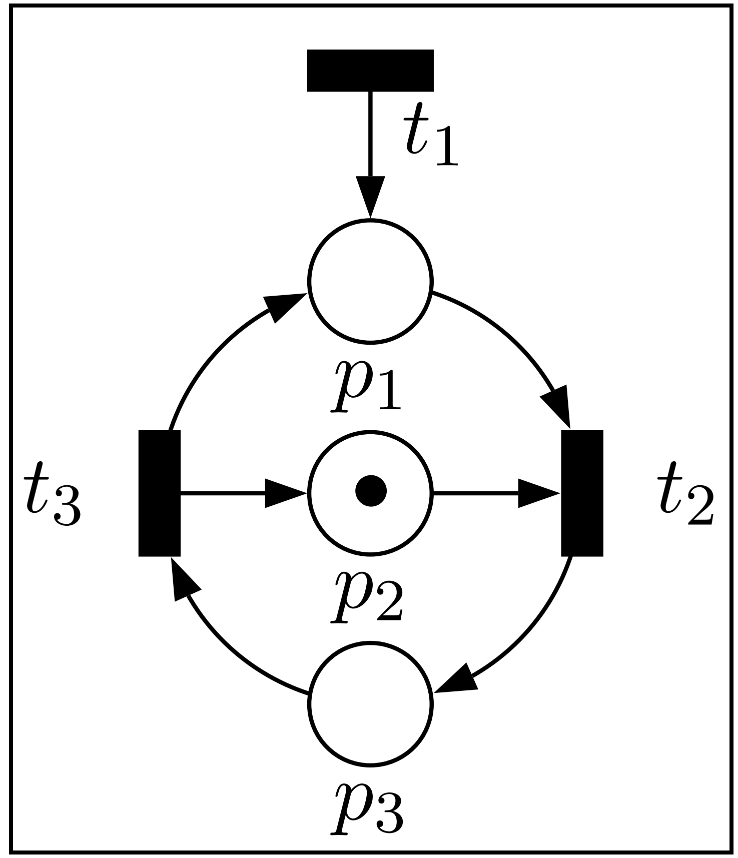
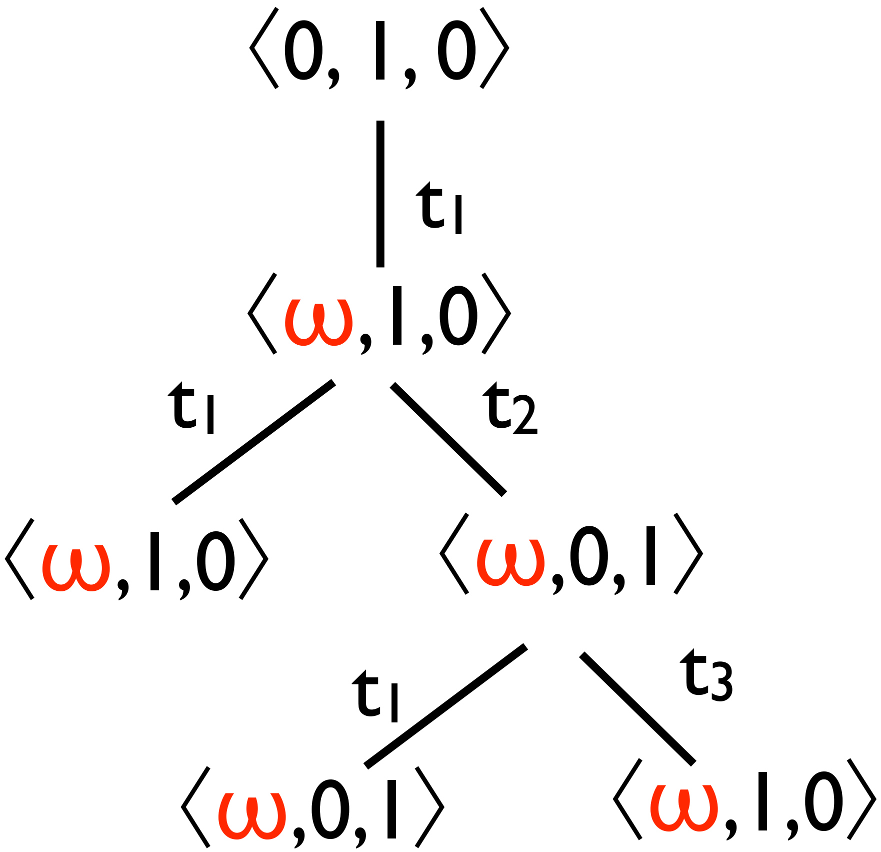
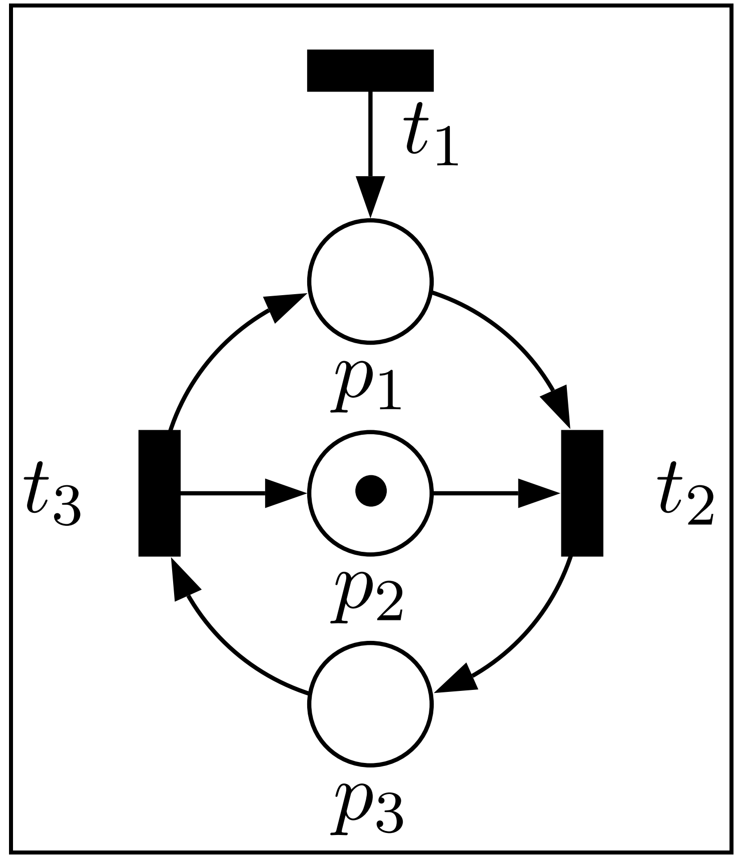



$t_2$ is semi-live

# Example

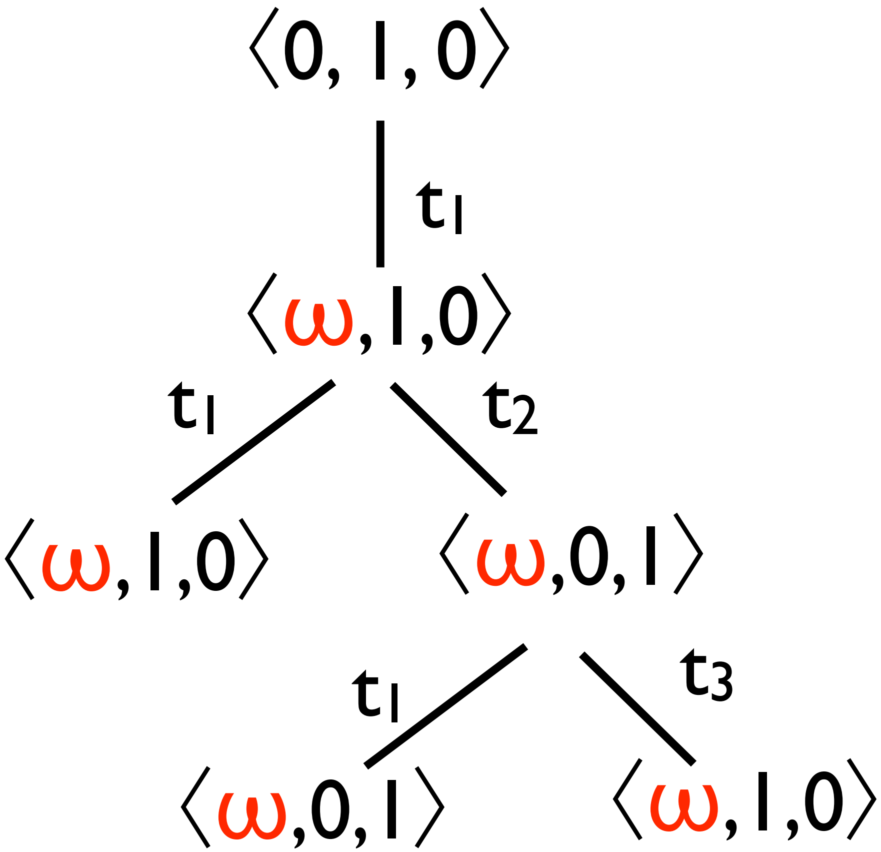


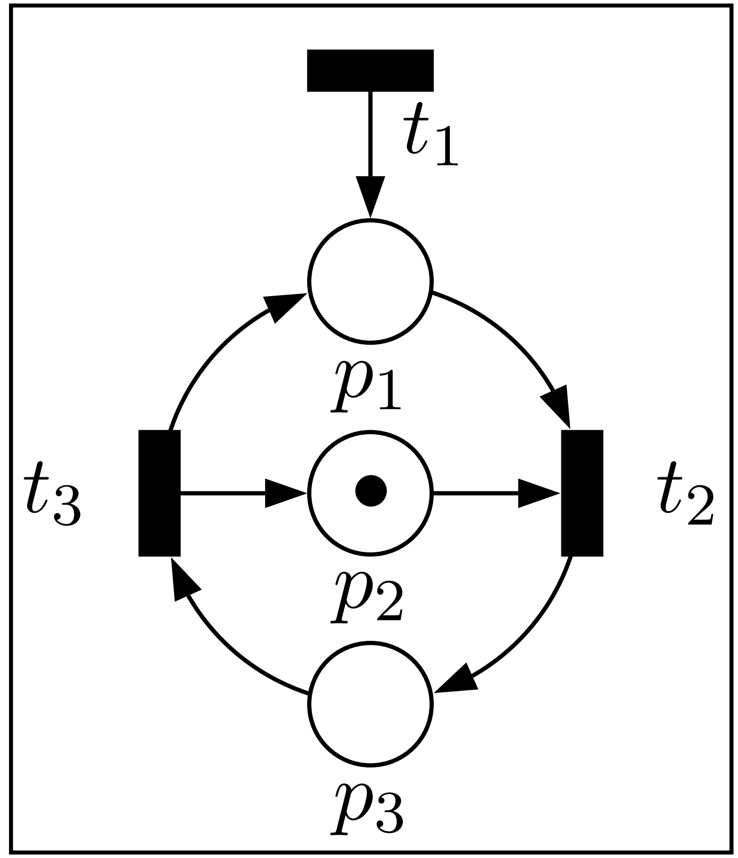




$t_2$ is semi-live

$p_2$ and $p_3$ are bounded

# Example

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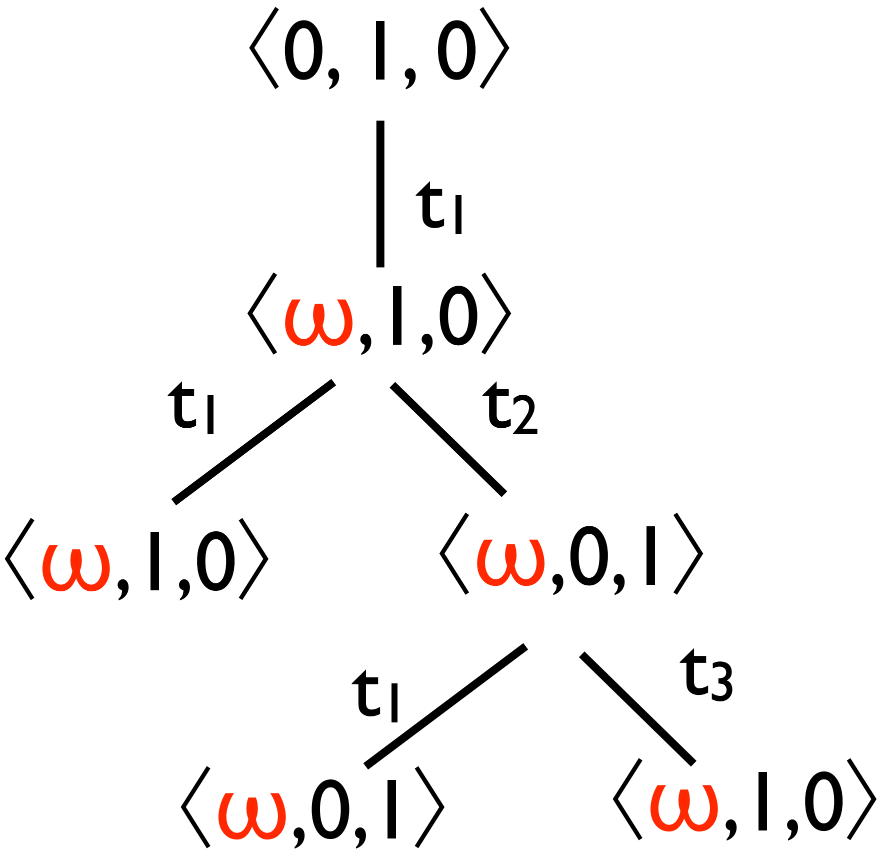


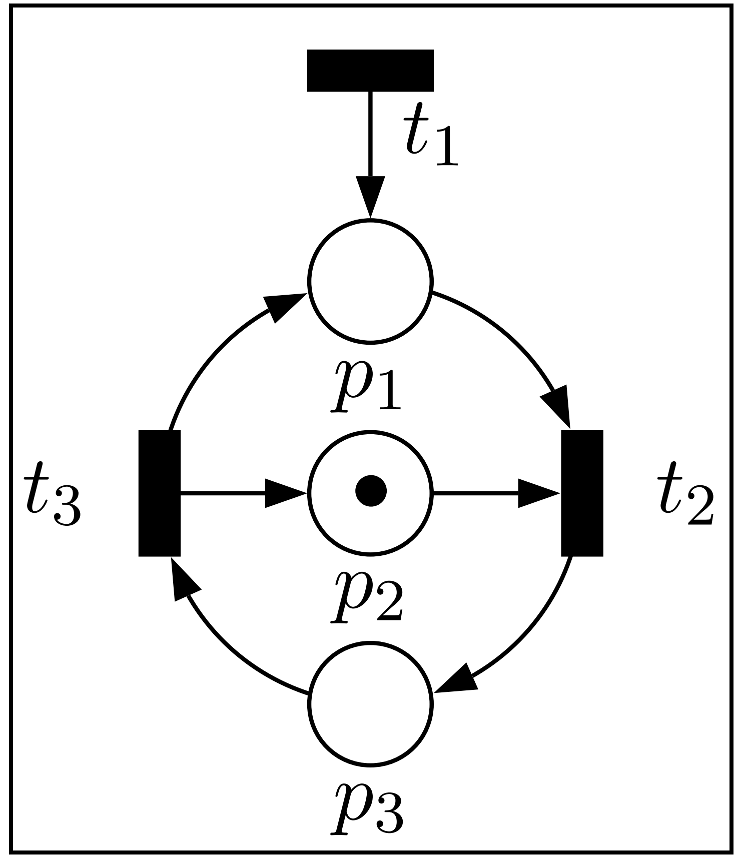


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The net is unbounded

# Coverability set

- Question: what is the relationship between:
  - the set of reachable markings and
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might be infinite

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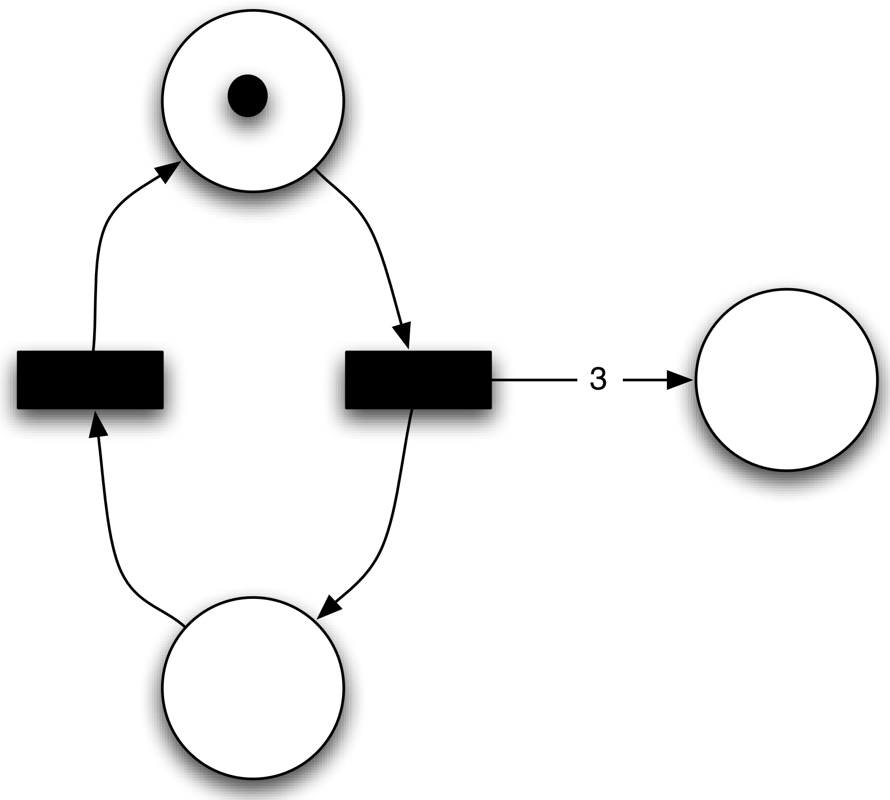

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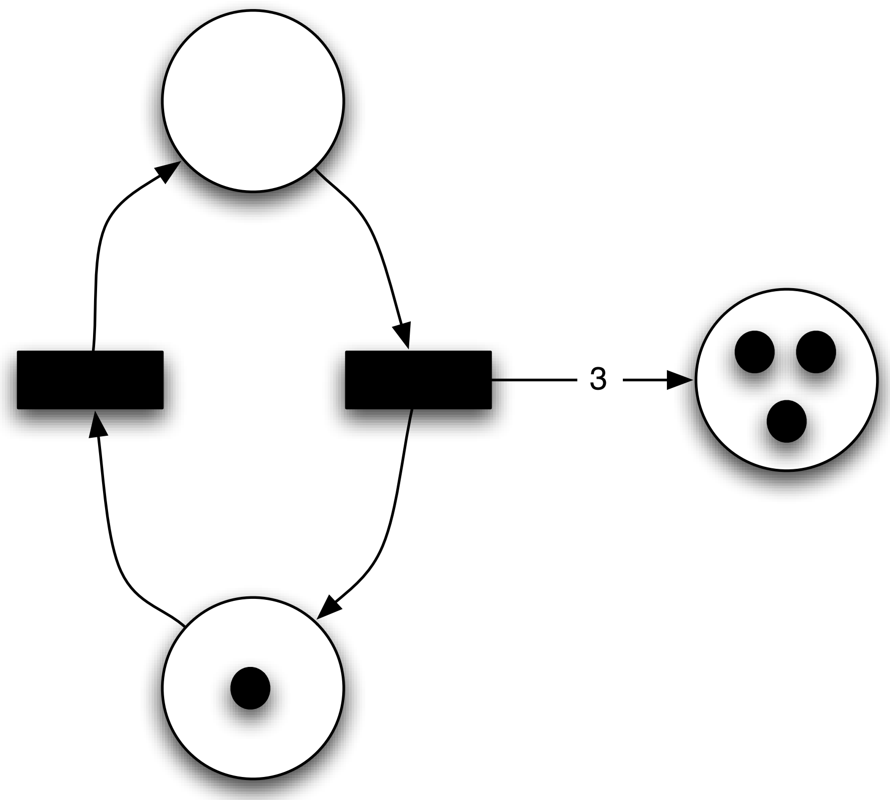

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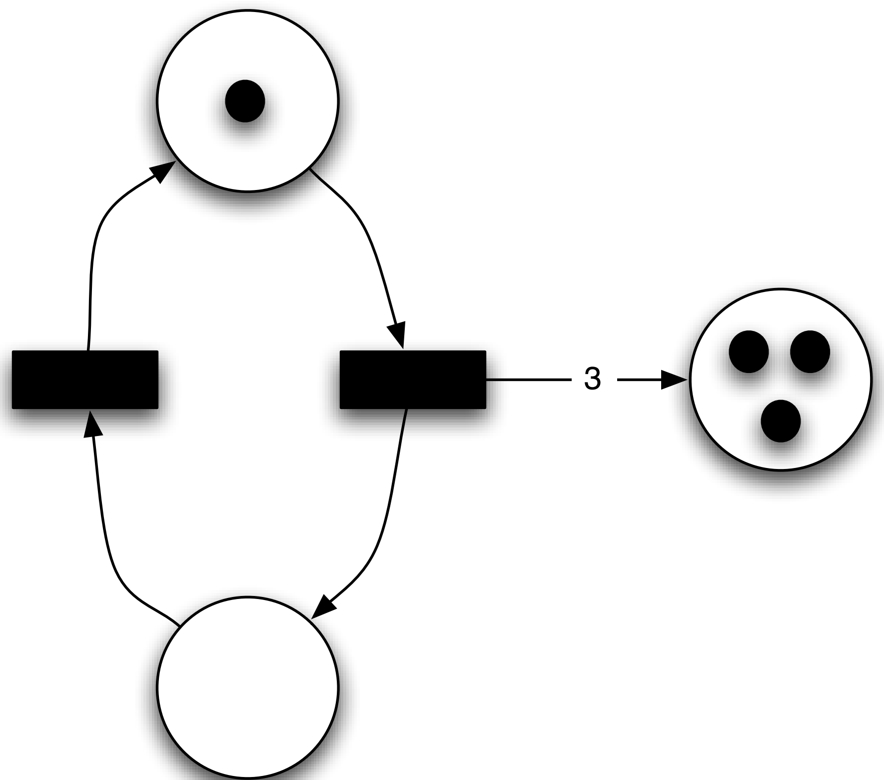

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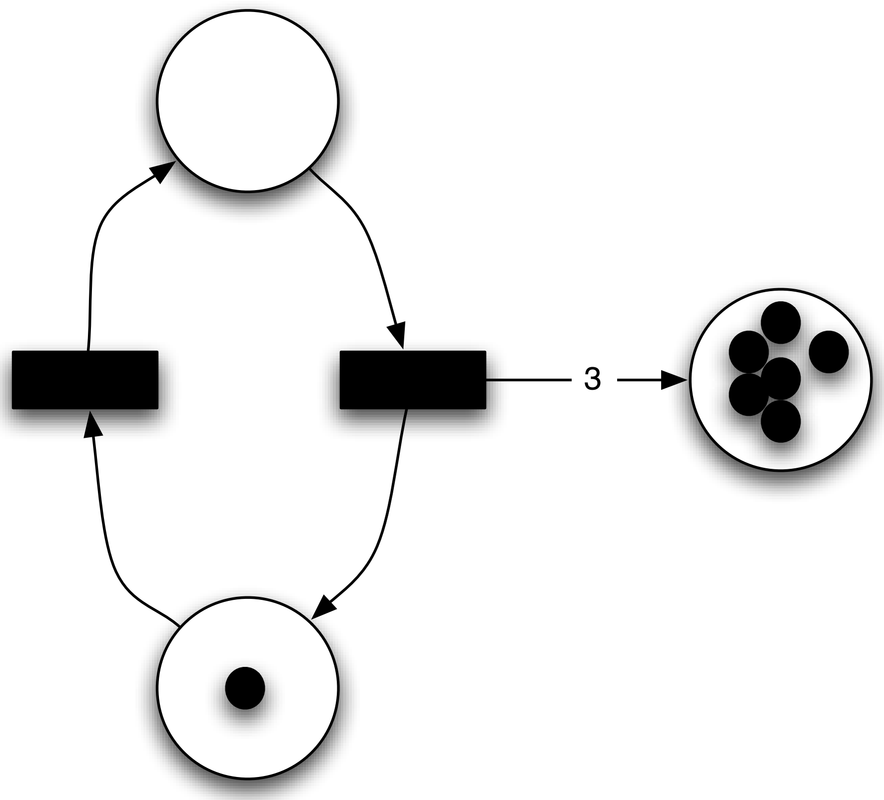


# Example

- Set of reachable markings:

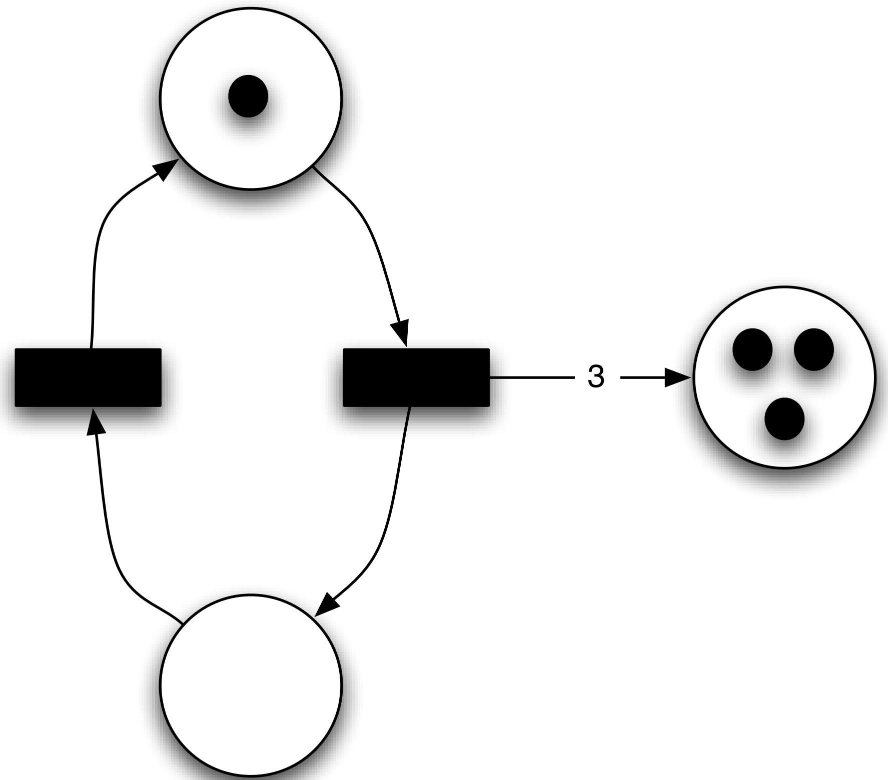
$\{ \langle 1, 0, 3.i \rangle \ , \ \langle 0, 1, 3.i \rangle \ \mid i \geqslant 0 \}$

- Set of nodes of the K&M tree:

$\{ \langle 1, 0, 0 \rangle \quad \langle 1, 0, \omega \rangle \ , \ \langle 0, 1, \omega \rangle \ \}$

- This set "represents":

$\{ \langle 1, 0, i \rangle \ , \ \langle 0, 1, i \rangle \ \mid i \geqslant 0 \}$



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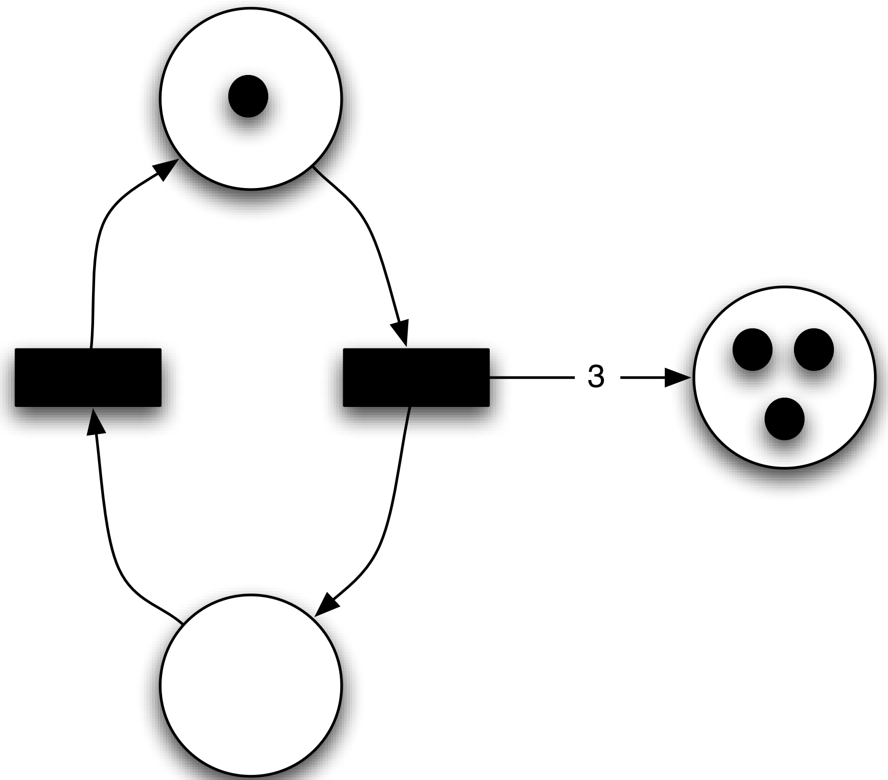


Clearly: $\neq$

# Example

Reach

$\{ \langle 1, 0, 3.i \rangle \ , \ \langle 0, 1, 3.i \rangle \ | \ i \geqslant 0 \}$

vs

K&M

$\{ \langle 1, 0, i \rangle \ , \ \langle 0, 1, i \rangle \ | \ i \geqslant 0 \}$

- Clearly, the K&M set contains more markings than the set of reachable markings:

$$\square \subseteq \square$$

- However, for every marking m in the K&M set, there exists a reachable marking m' s.t.:

$$m' \succcurlyeq m$$

# Example

Reach

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$$m' \succcurlyeq m$$

$$= \quad + \{m \mid \text{there is } m' \text{ in} \quad \text{with } m' \succcurlyeq m\}$$

# Downward-closure

- Let us assume that any natural number i is s.t.

$$i < \omega$$

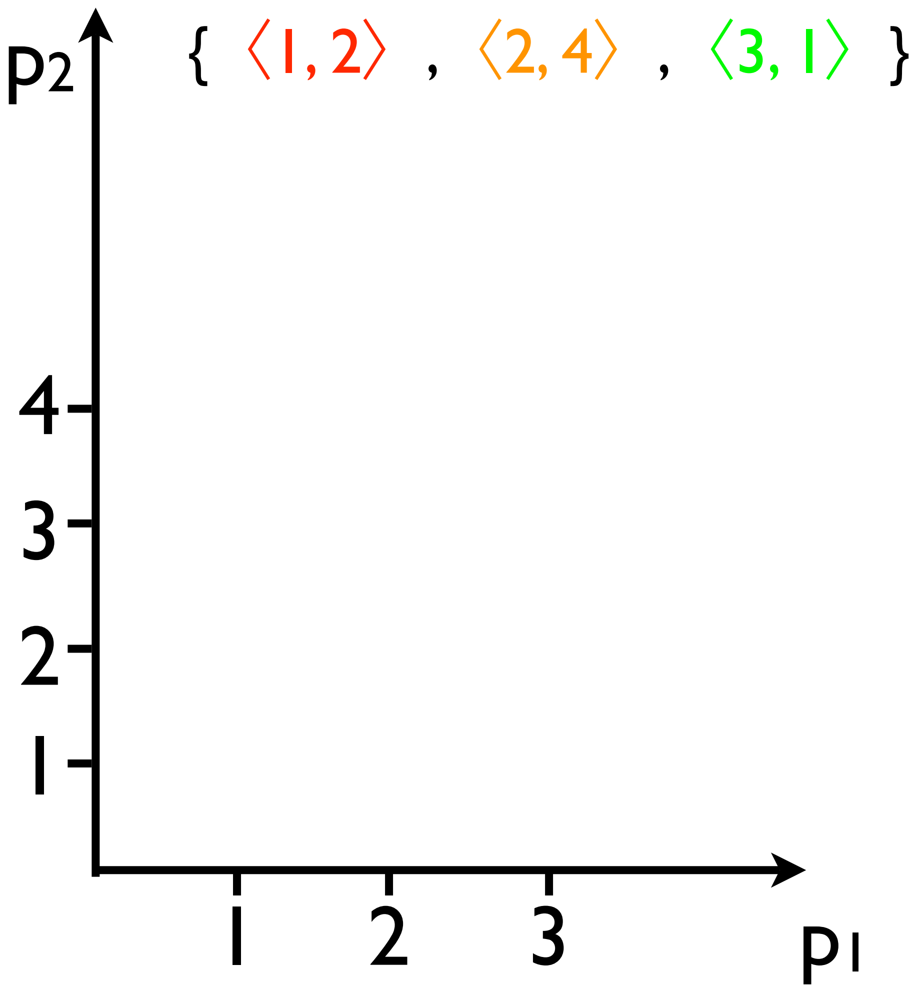
- Let $m$ be a marking (possibly with $\omega$), then its downward-closure is the set:

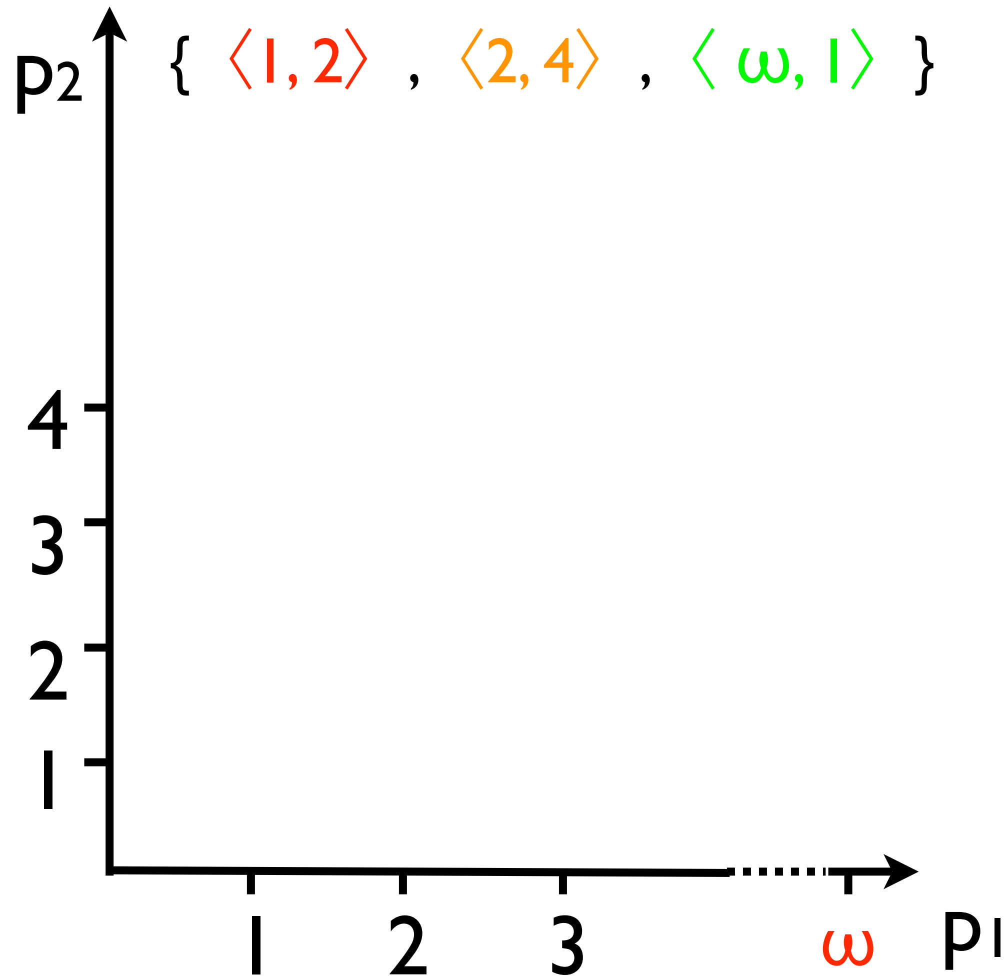
$$\downarrow m = \{m' \mid m' \preccurlyeq m\}$$

- Let $S=\{m_1, m_2, \ldots m_k\}$ be a set of markings, then:

$$\downarrow S = \downarrow m_1 \cup \downarrow m_2 \cup \ldots \cup \downarrow m_k$$

# Examples in 2 dim.



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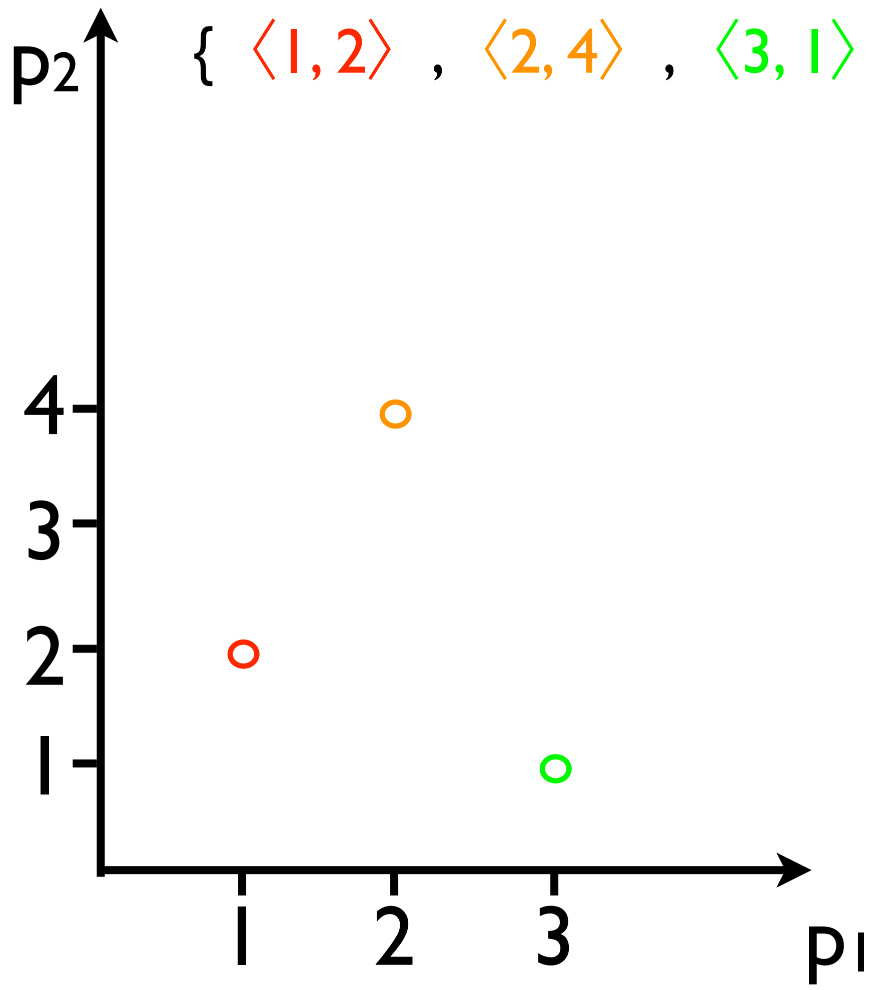


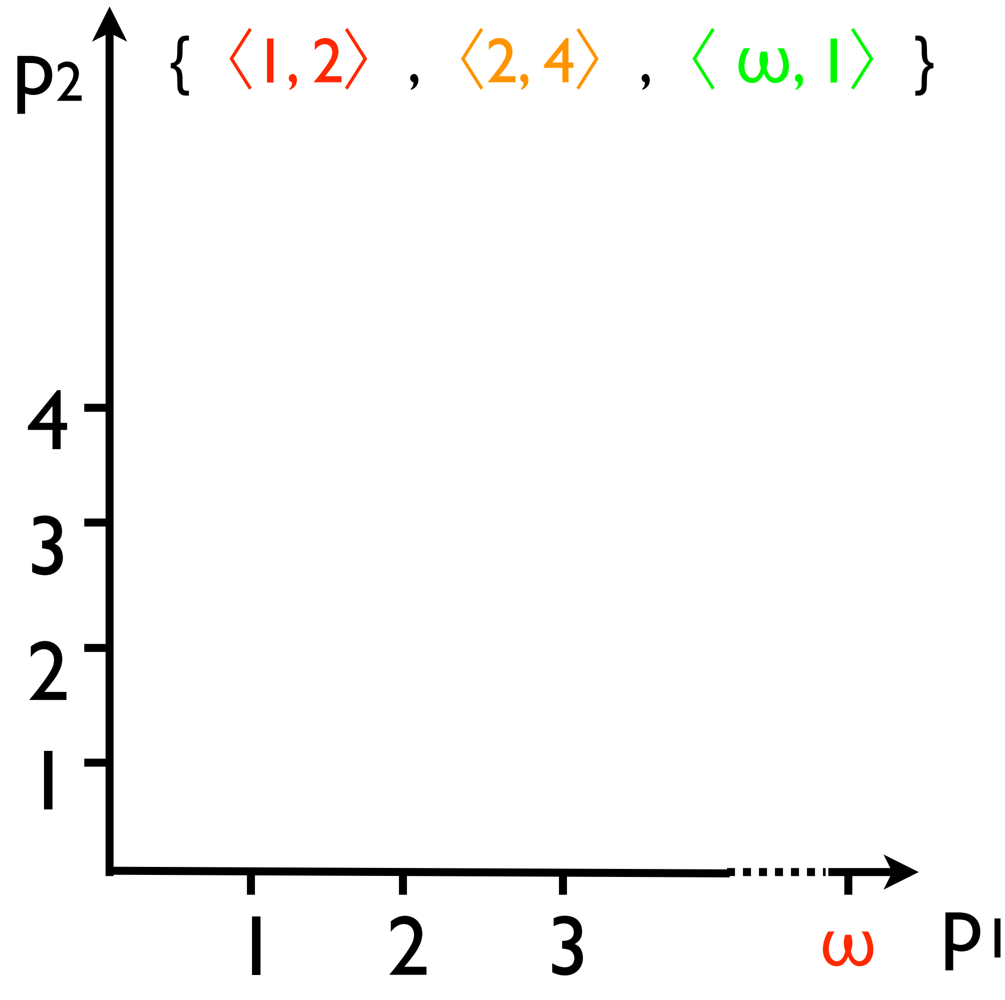

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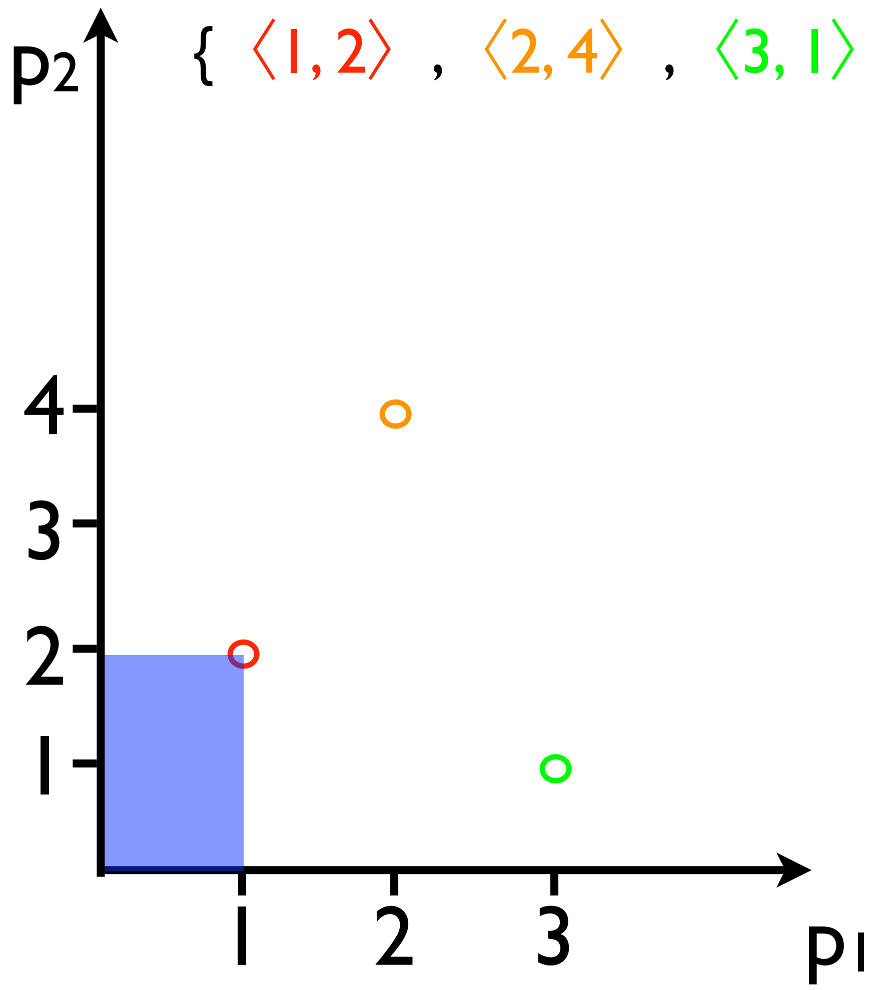


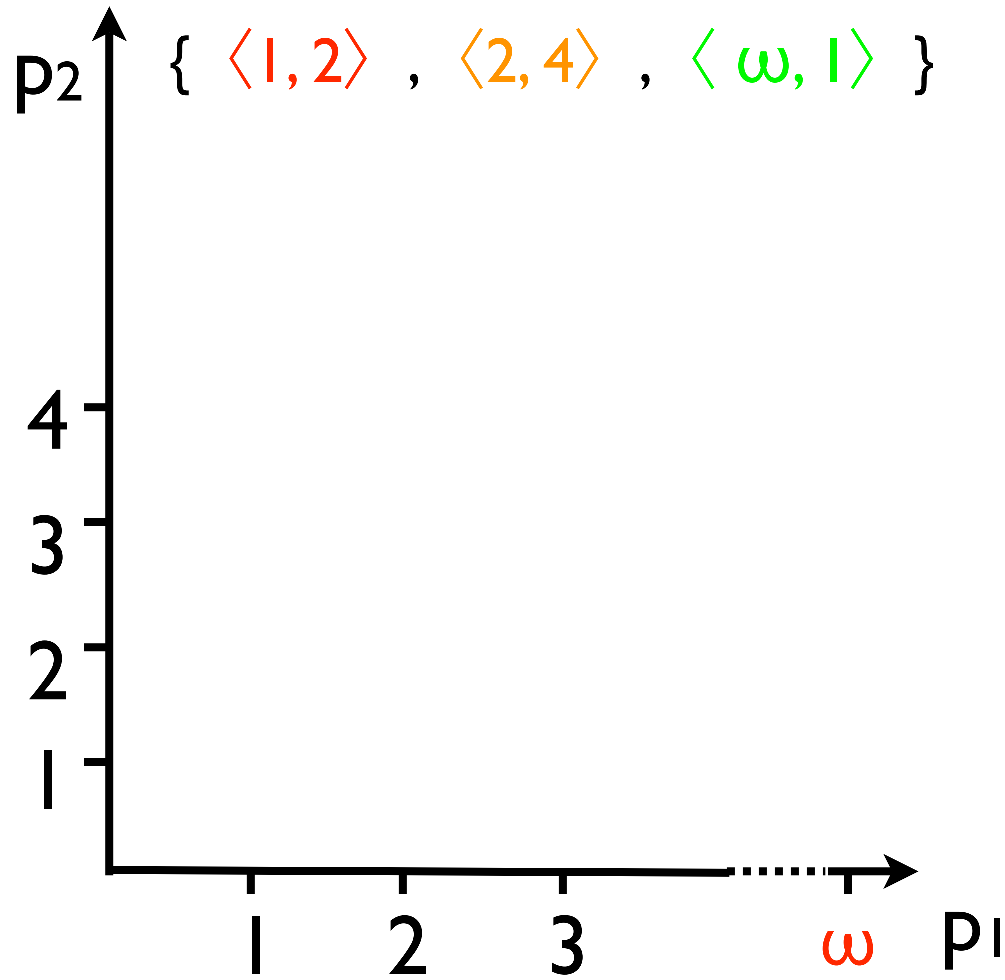

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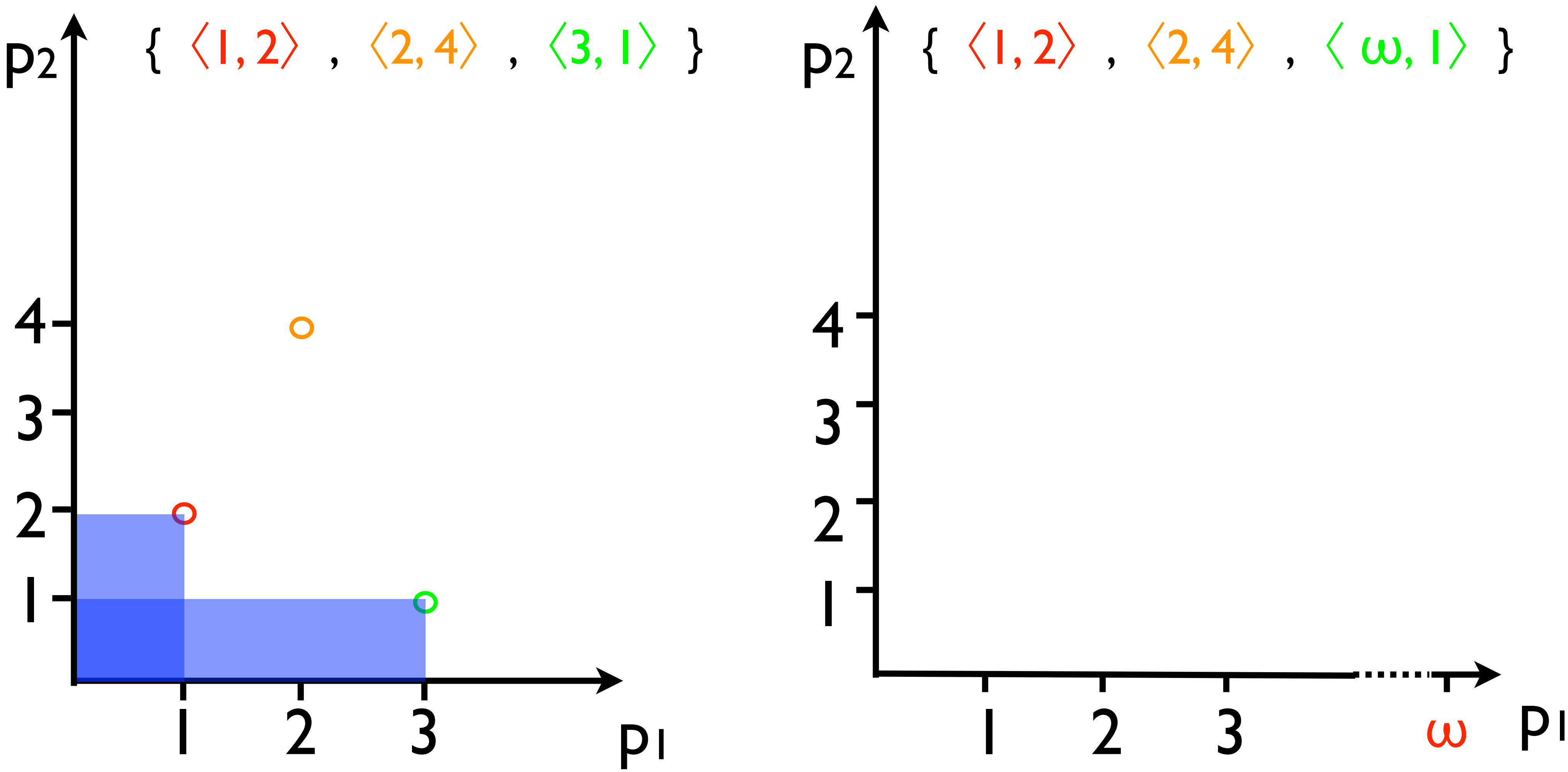
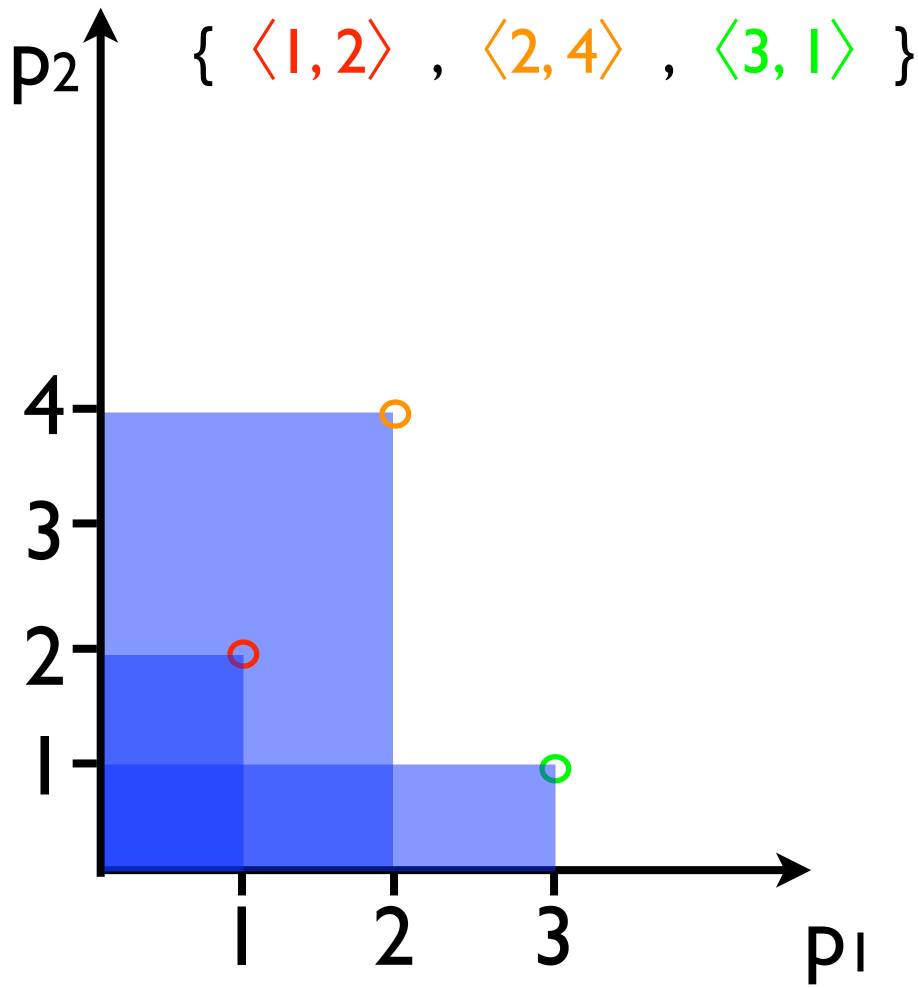
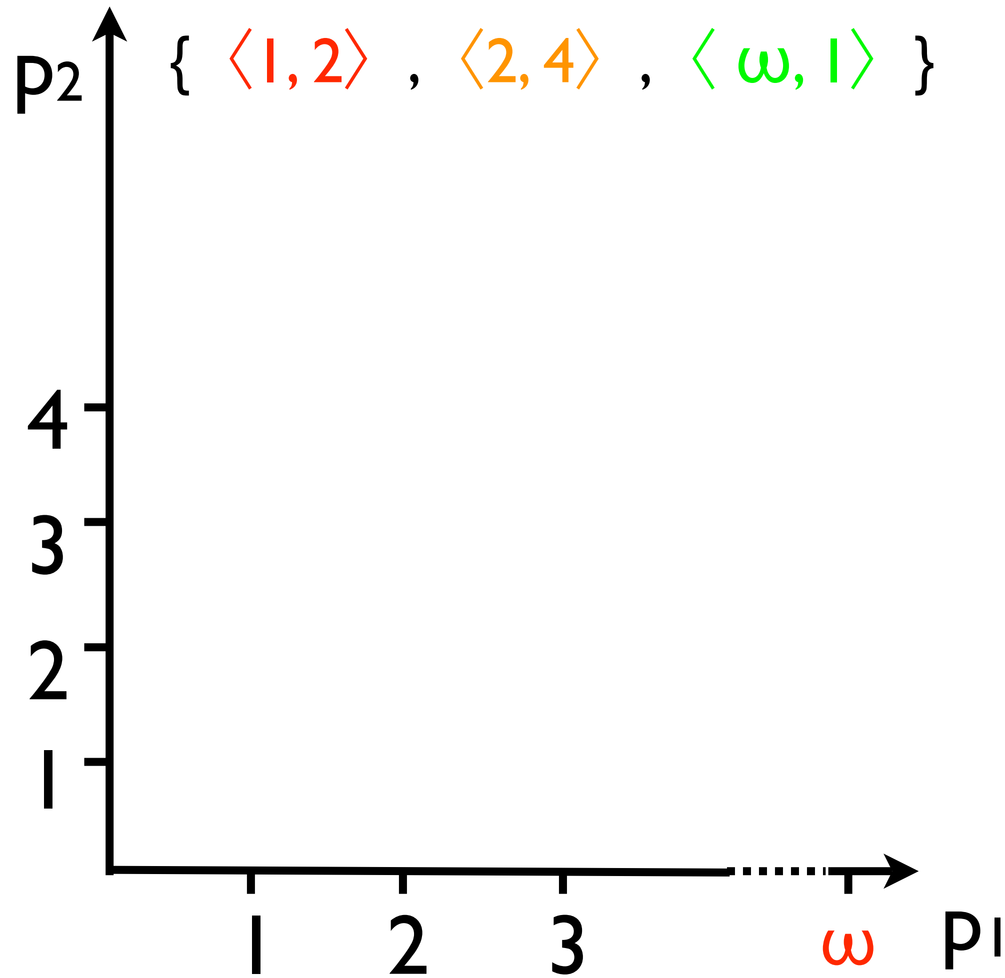


# Examples in 2 dim.



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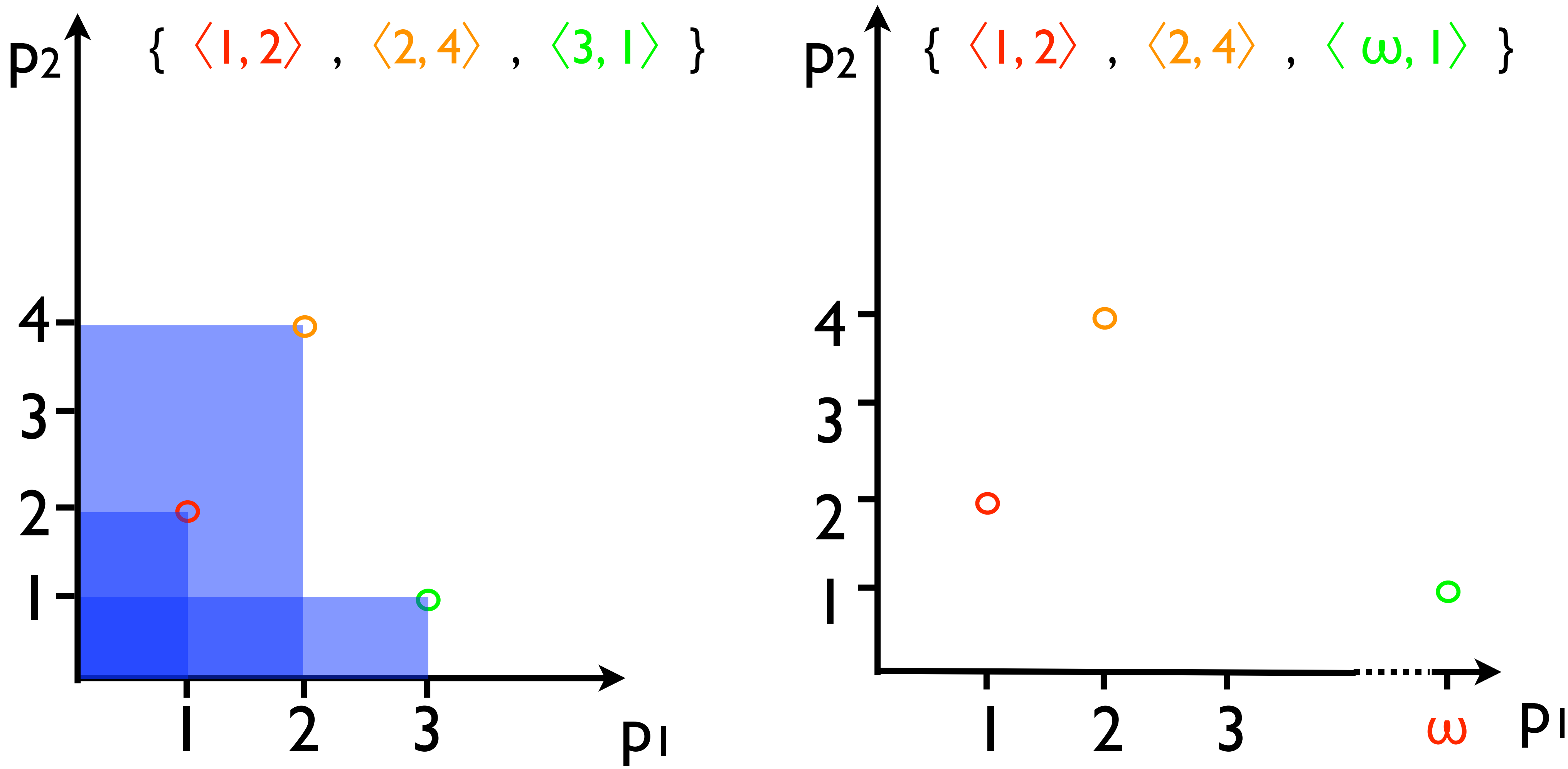

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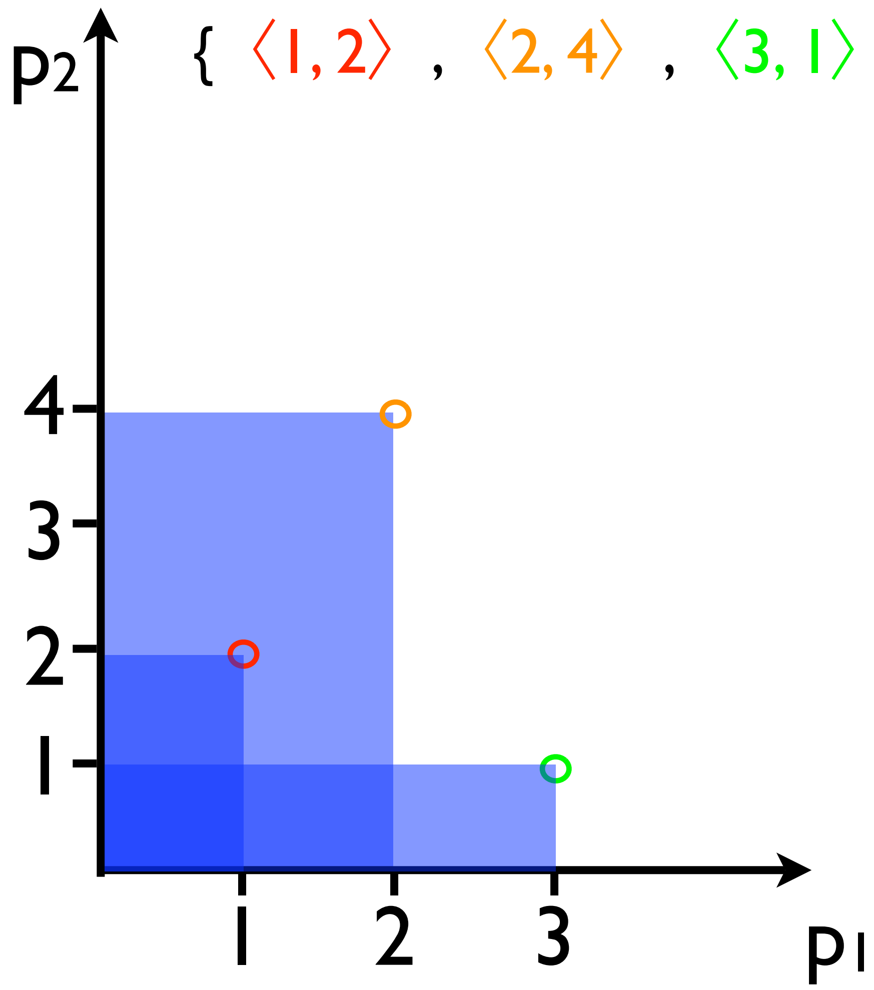


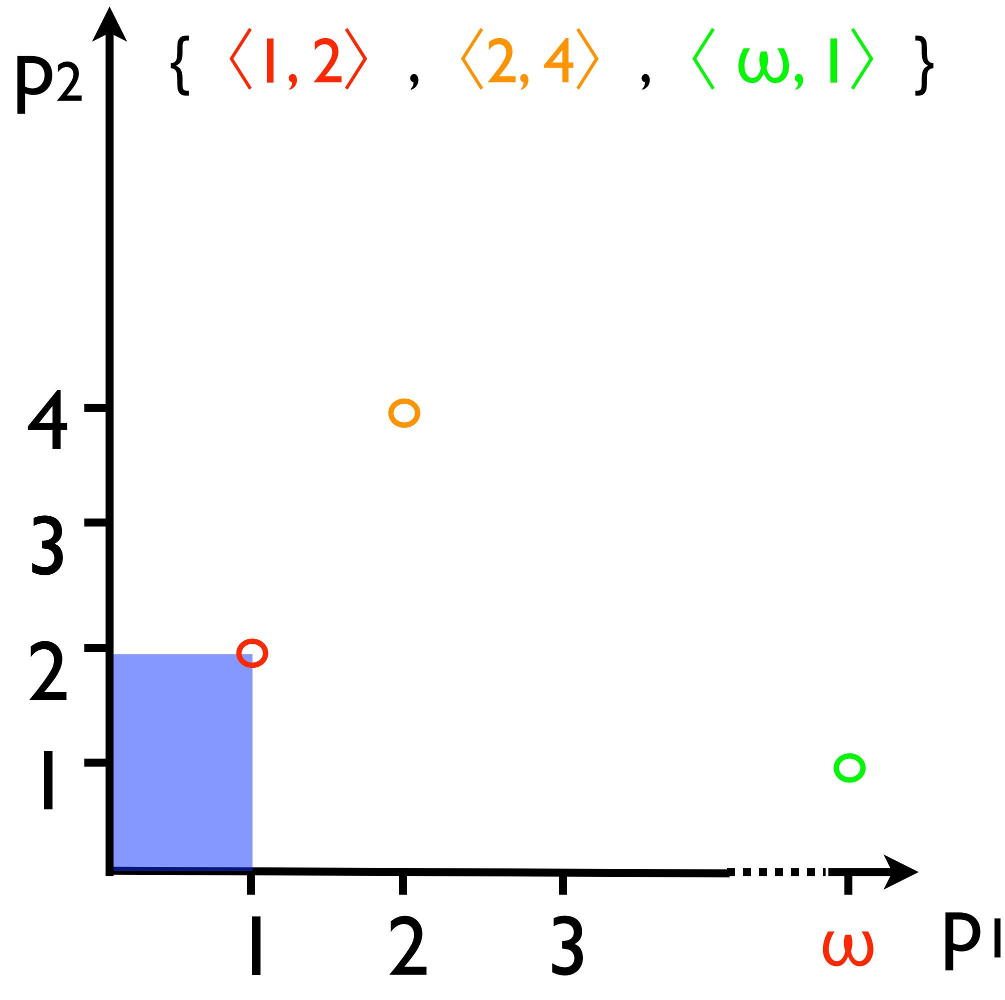

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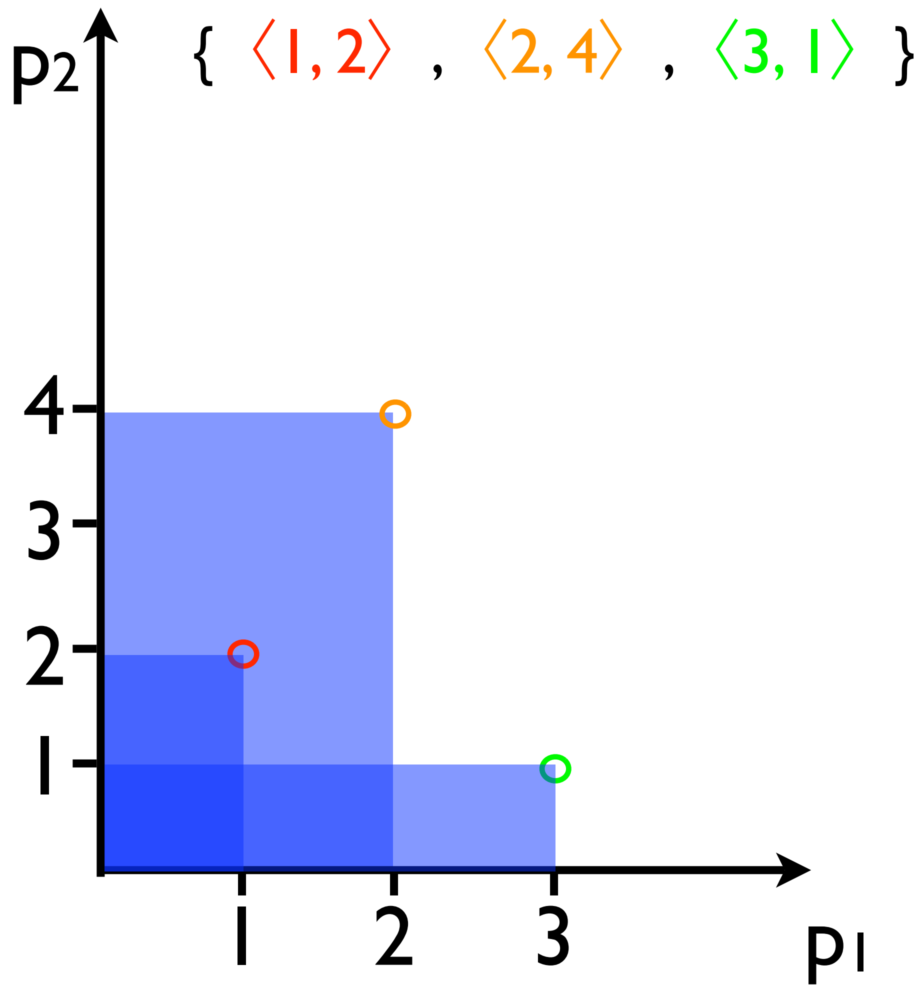


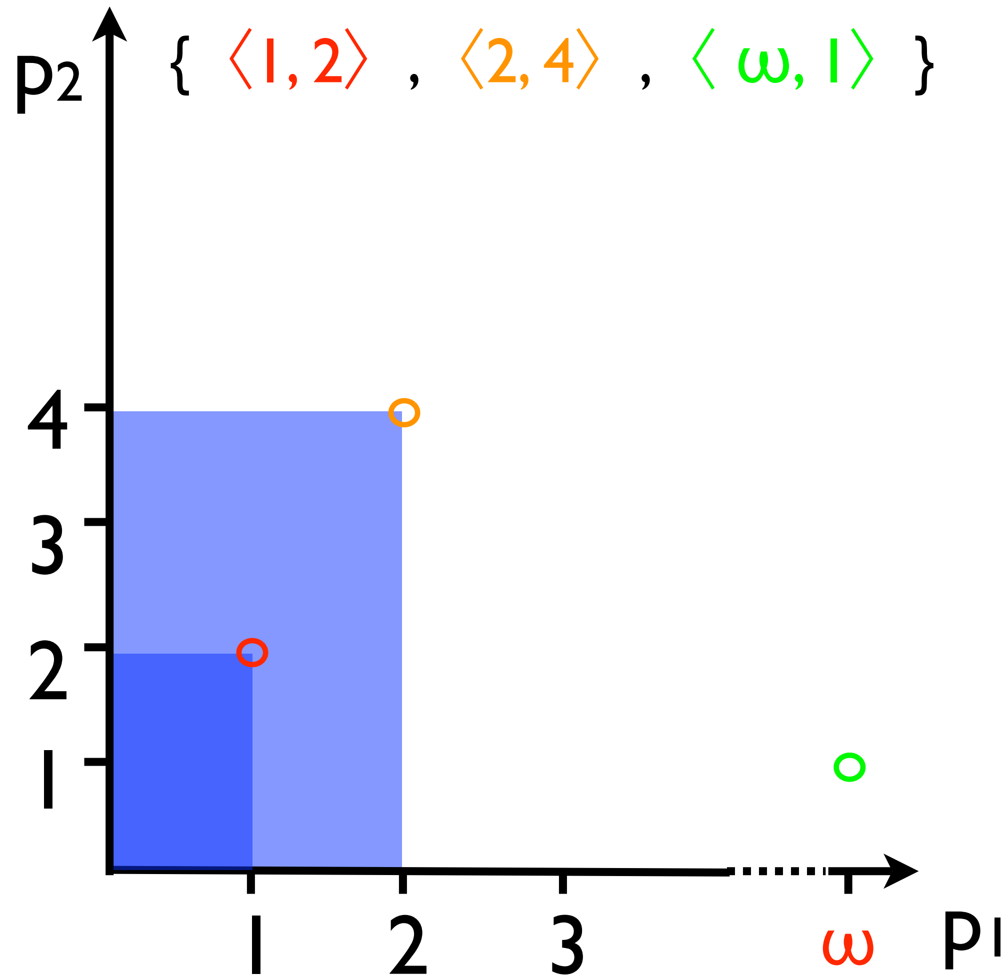

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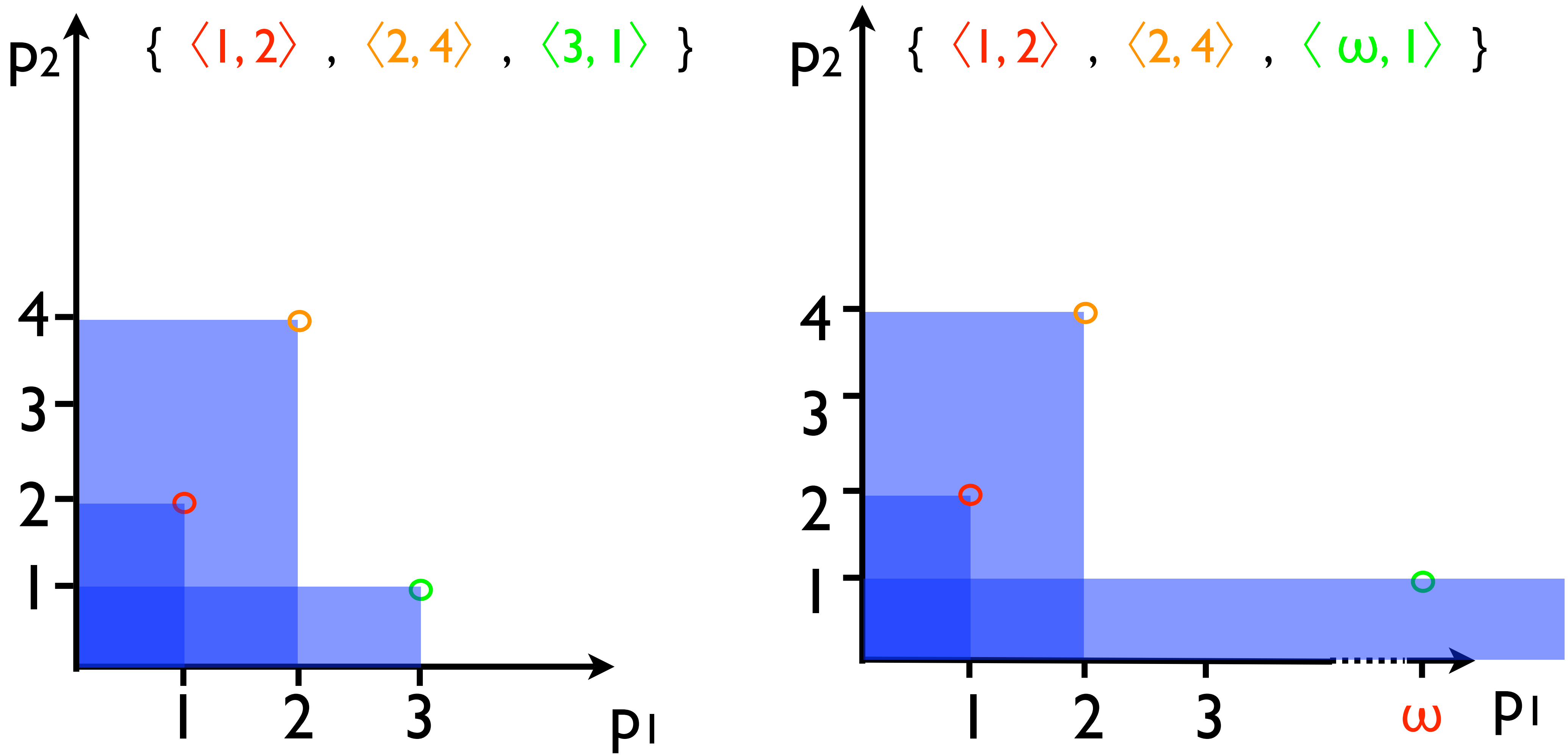


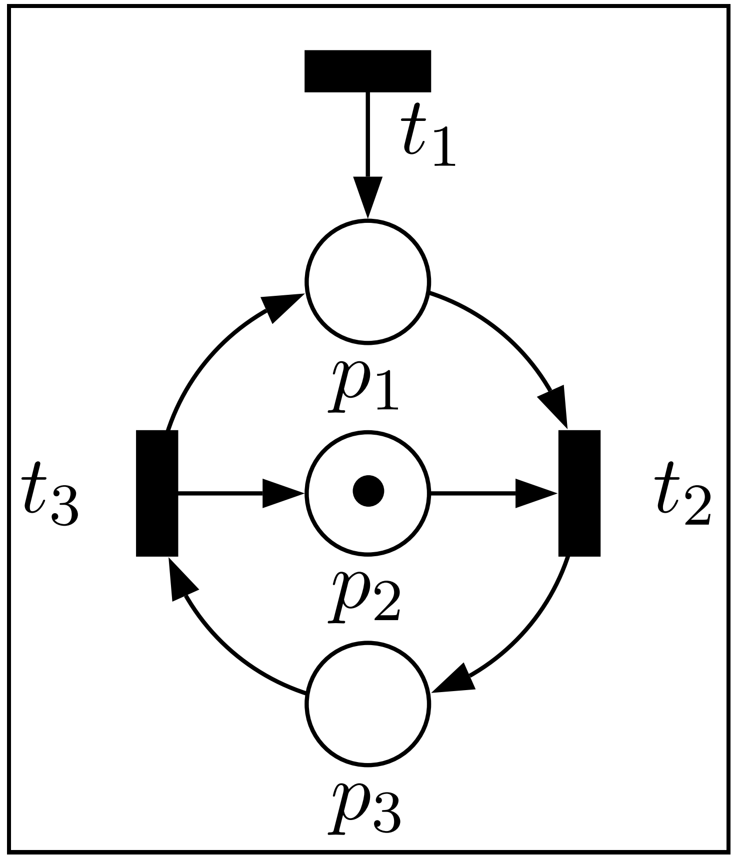
# Properties of the K&M tree

- The set of all the markings that appear in a K&M tree is called a coverability set of the net.
  - Notation: Cover(N)
- Theorem: $\downarrow$ Cover(N) = $\downarrow$ Reach(N)
- Theorem: Reach(N) $\subseteq$ $\downarrow$ Cover(N)
- Hence, $\downarrow$ Cover(N) is a finite over-approximation of Reach(N)

# Example

$$\mathrm{Reach}(N) = \{\ \langle i, 1, 0\rangle\ ,\ \langle i, 0, 1\rangle\ \mid i \geq 0\ \}$$

$$\mathrm{Cover}(N) = \downarrow\{\ \langle \omega, 1, 0\rangle\ ,\ \langle \omega, 0, 1\rangle\ \} = \mathrm{Reach}(N)\ \cup\ \{\ \langle 0, 0, 0\rangle\ \}$$




# Advertisement


- Recently, we have defined a new algorithm to compute the coverability set of a Petri net.

- It is several order of magnitudes more efficient than K&M

| Example | | | | | KM | | CovProc | | |
|---|---|---|---|---|---|---|---|---|---|
| Name | P | T | MCS | Tp | Nodes | Time | Max P. | Tot. P. | Time |
| CSM | 14 | 13 | 16 | U | $>2.40 \cdot 10^6$ | $\times$ | 178 | 248 | 0.34 |
| FMS | 22 | 20 | 24 | U | $>6.26 \cdot 10^5$ | $\times$ | 477 | 866 | 2.10 |
| PNCSA | 31 | 36 | 80 | U | $>1.02 \cdot 10^6$ | $\times$ | 2,617 | 13,408 | 113.79 |
| multipoll | 18 | 21 | 220 | U | $>1.16 \cdot 10^6$ | $\times$ | 14,034 | 14,113 | 365.90 |
| mesh2x2 | 32 | 32 | 256 | U | $>8.03 \cdot 10^5$ | $\times$ | 10,483 | 12,735 | 330.95 |

# The coverability problem

# Reachability: a natural question

- The reachability problem: given a marking $m$ is it reachable from $m_0$ ?

$m_0$

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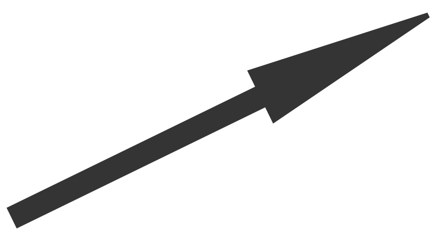

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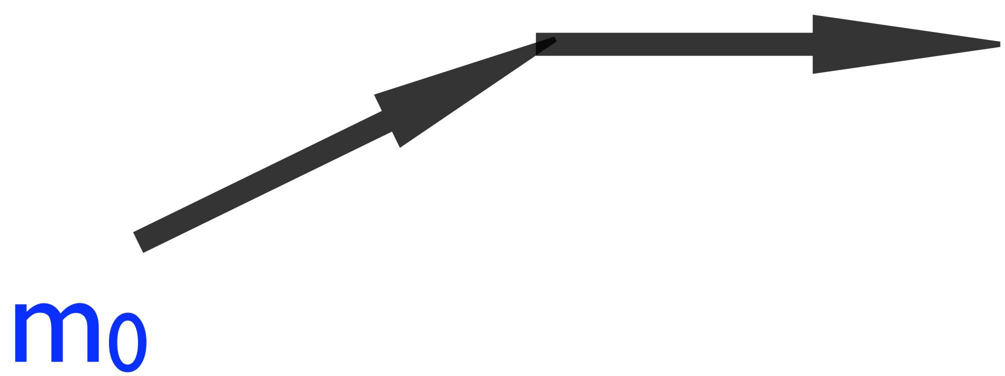

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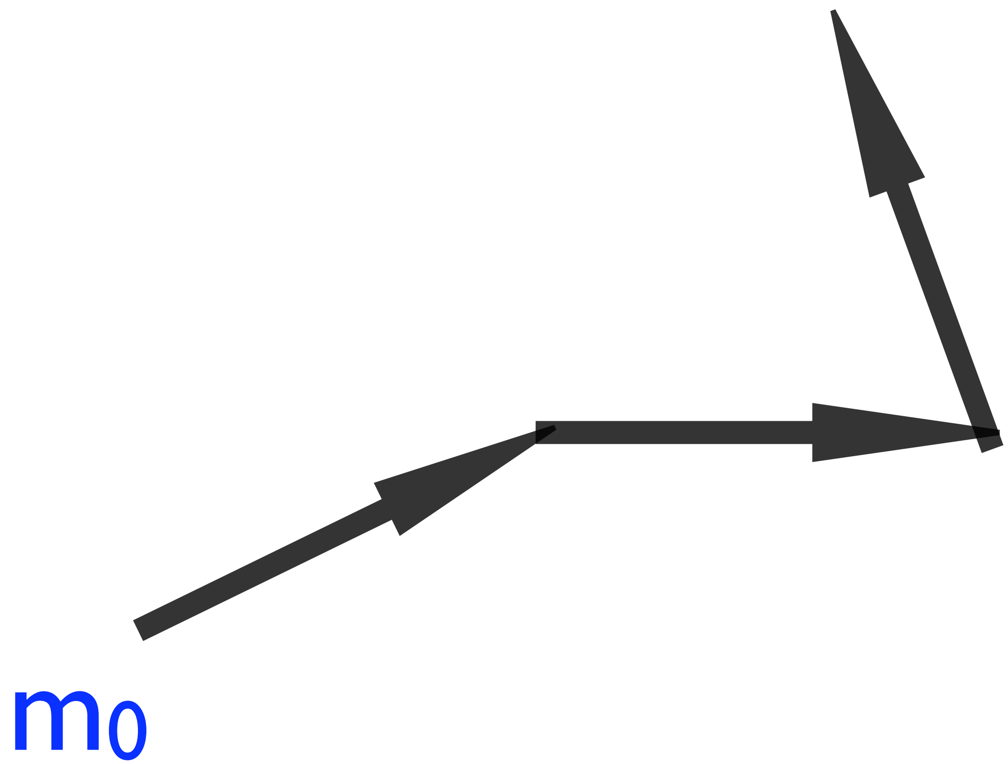

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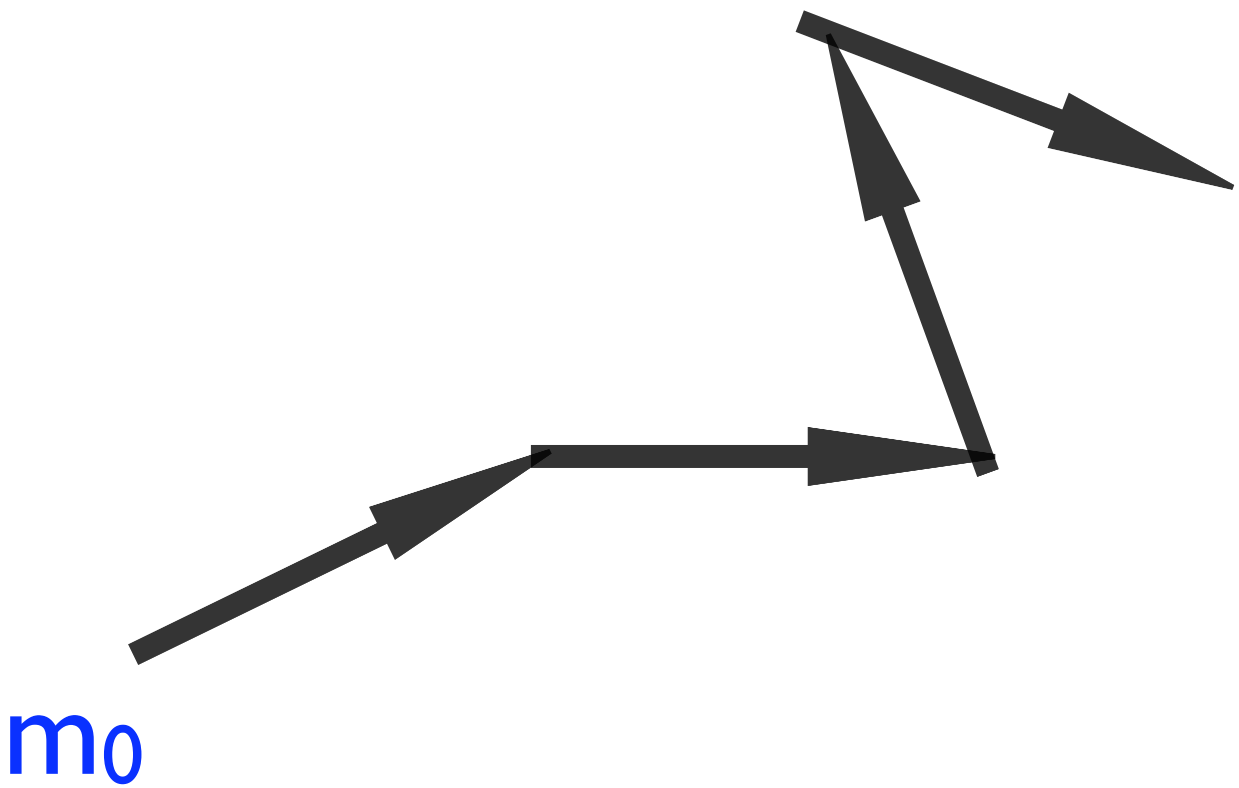

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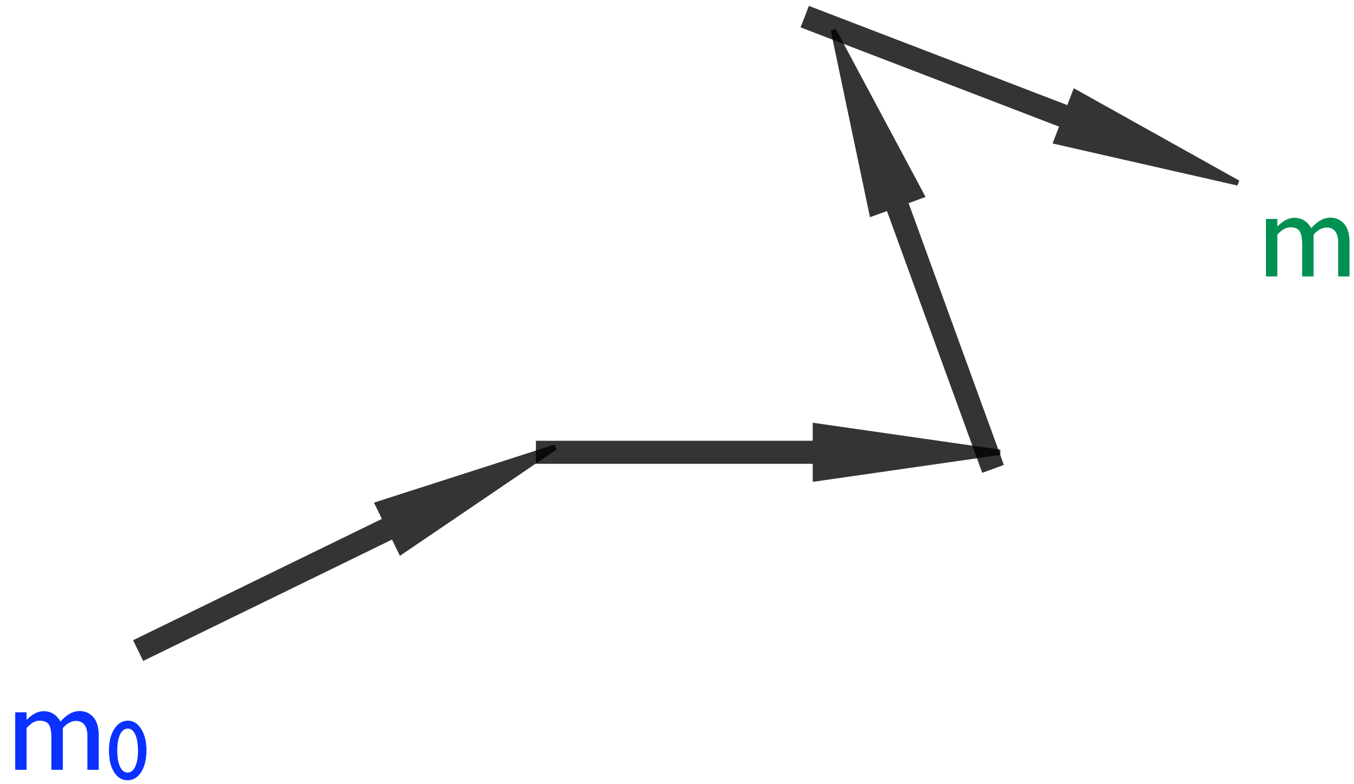
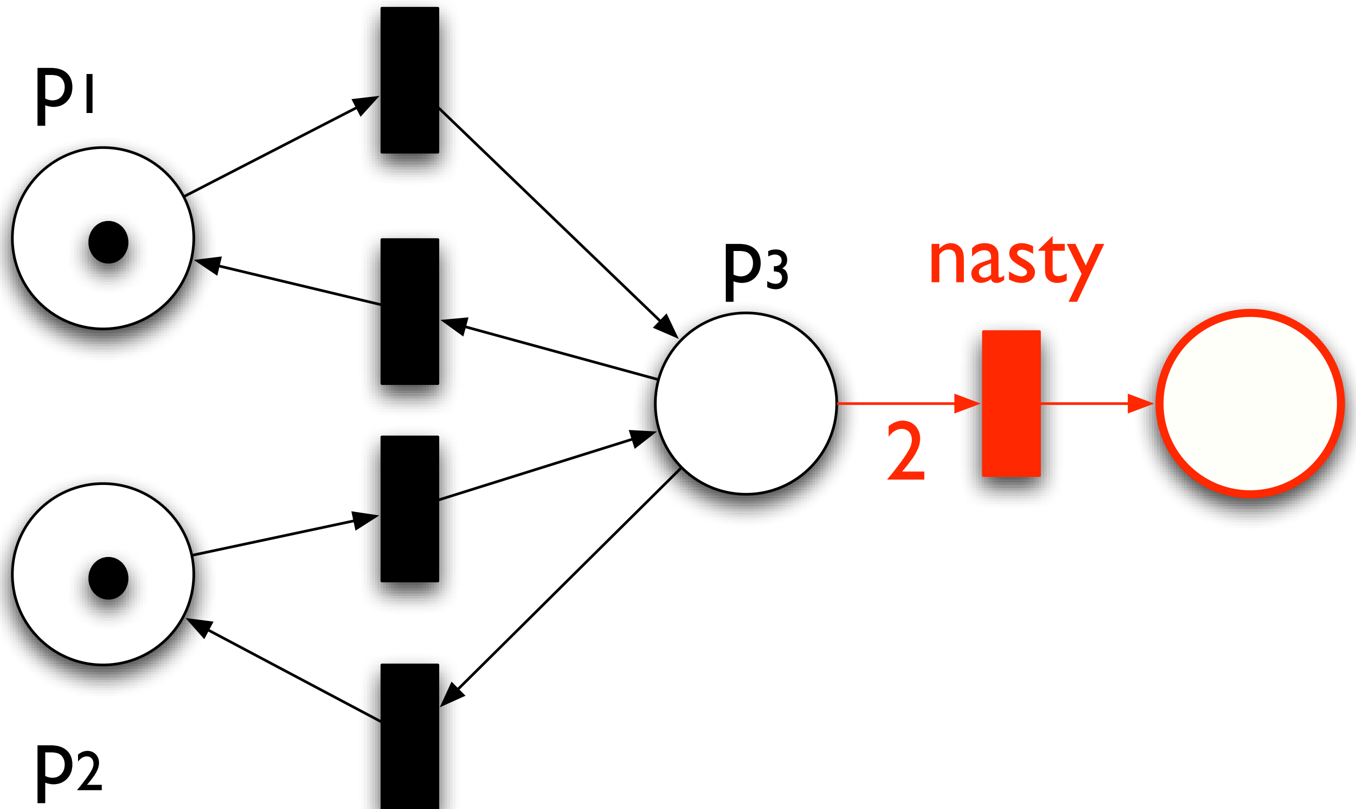


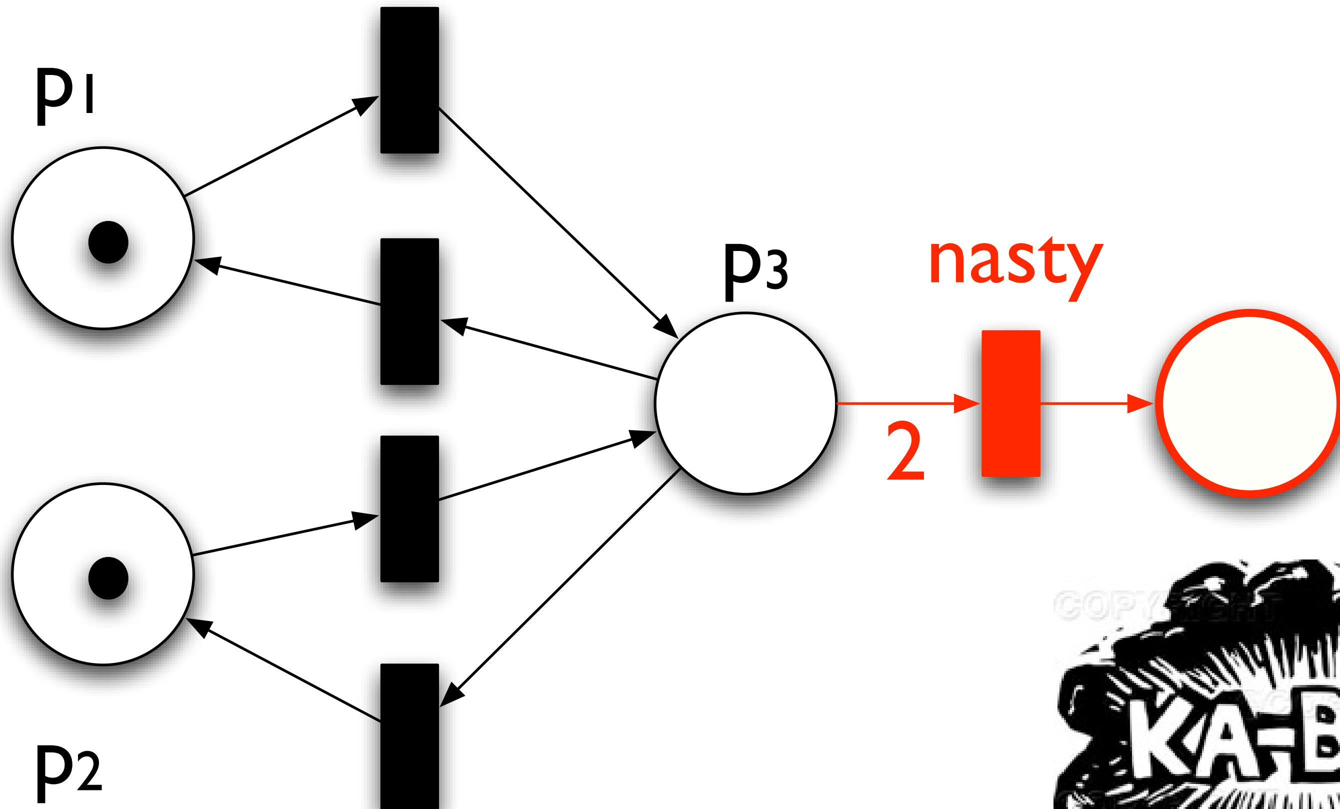
# Reachability: a natural question ??

- In the case  of Petri nets, asking whether a given marking is reachable does not always make sense...
- ... because Petri nets are monotonic

# Example



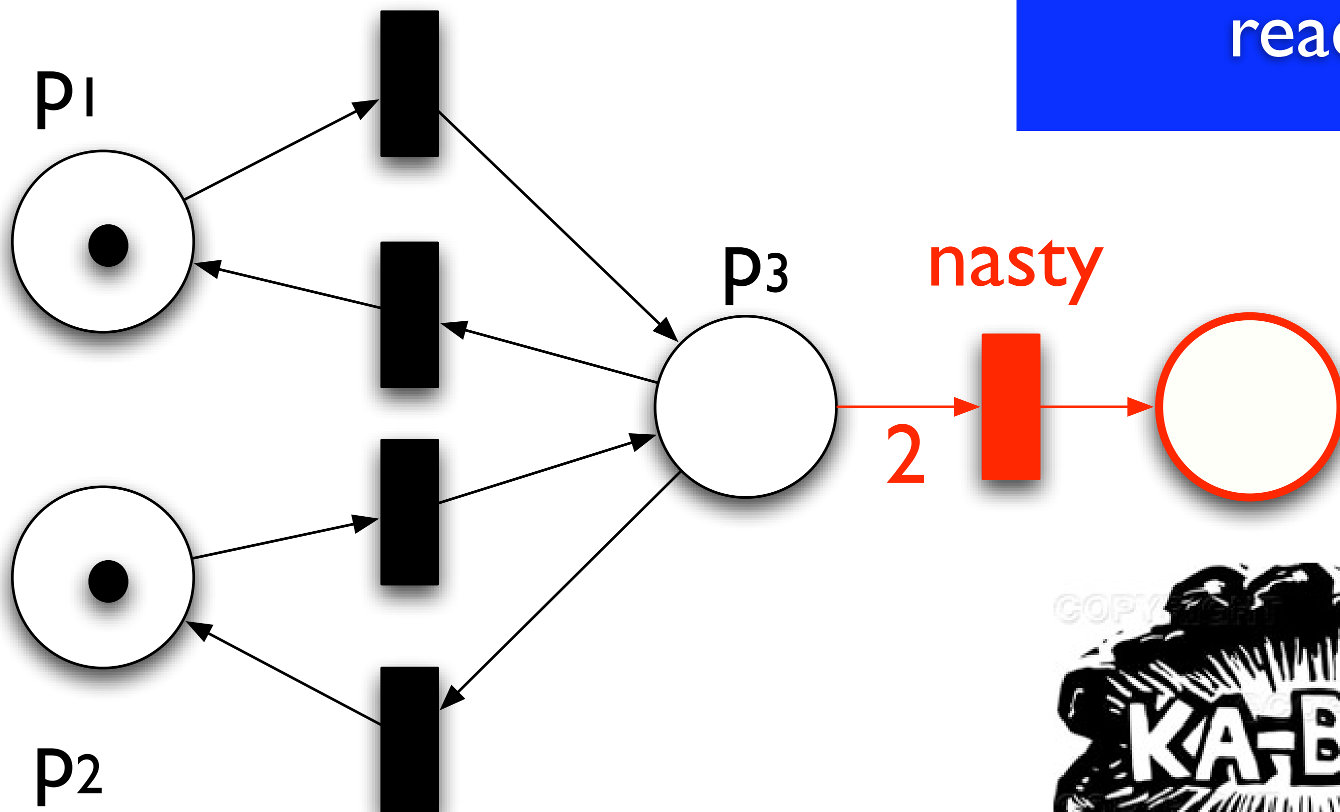
# Example




# Example

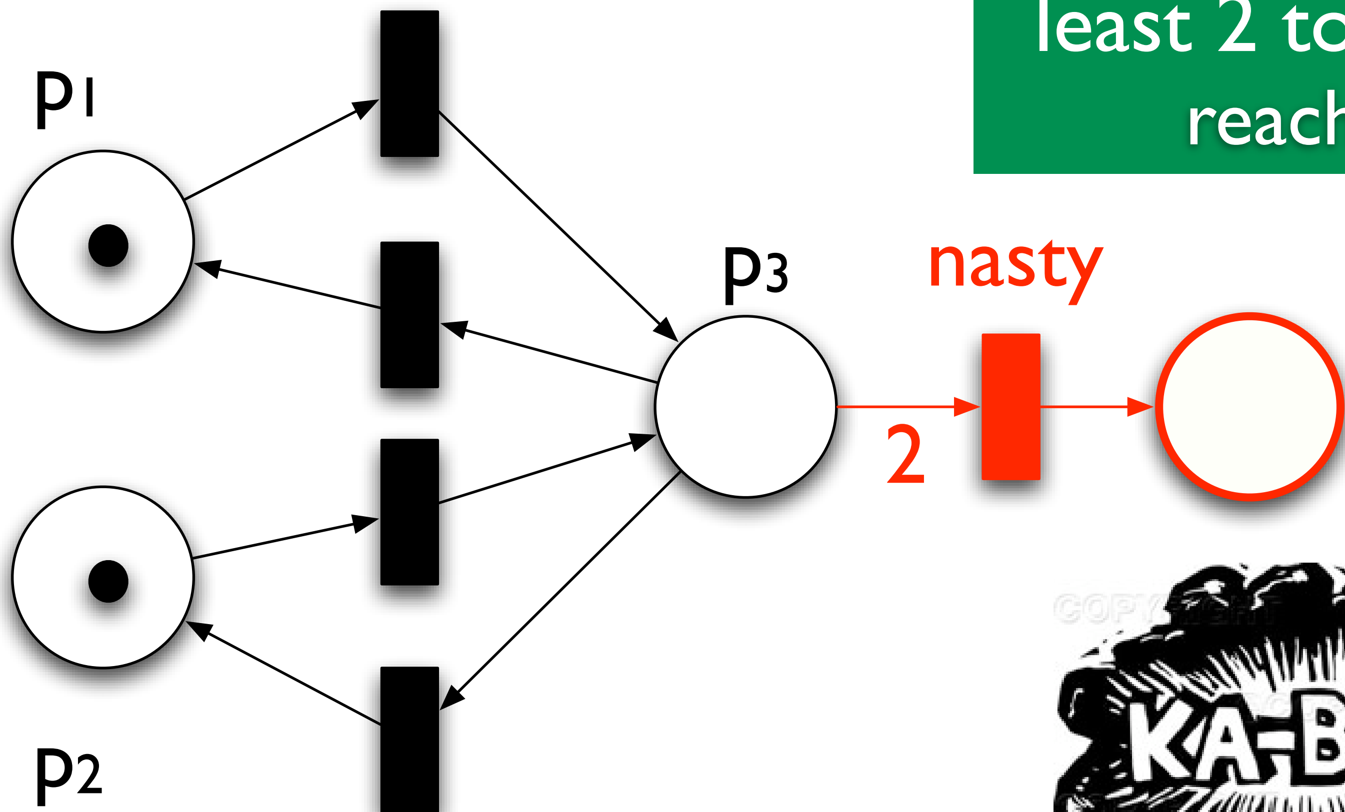
**Question**

is $\langle 0, 0, 2, 0 \rangle$ reachable ?



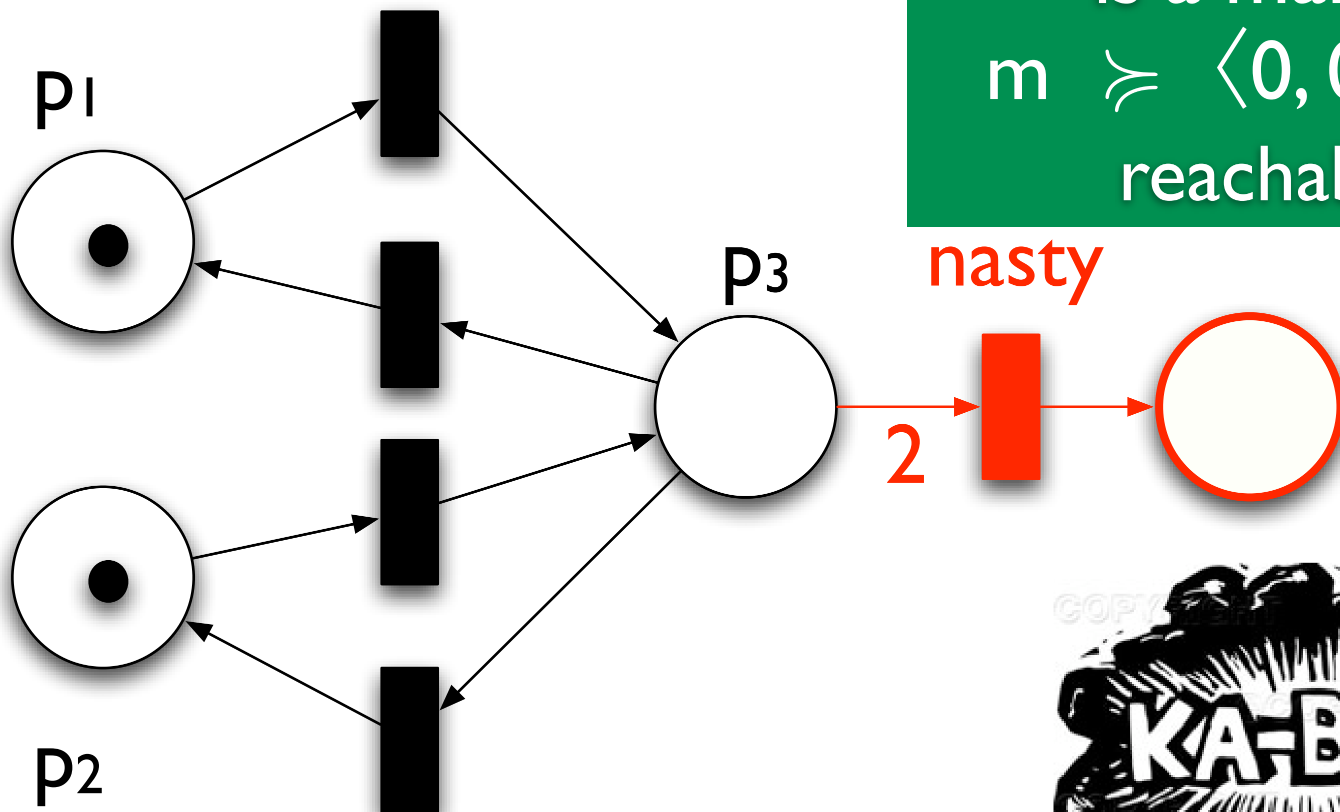

# Example

Better question
is a marking with at least 2 tokens in p3 reachable ?




# Example

Better question
is a marking
$m \succcurlyeq \langle 0, 0, 2, 0 \rangle$
reachable ?




# The coverability problem

Does there exist a reachable marking which is larger than some marking b ?

$b$

$m_0$

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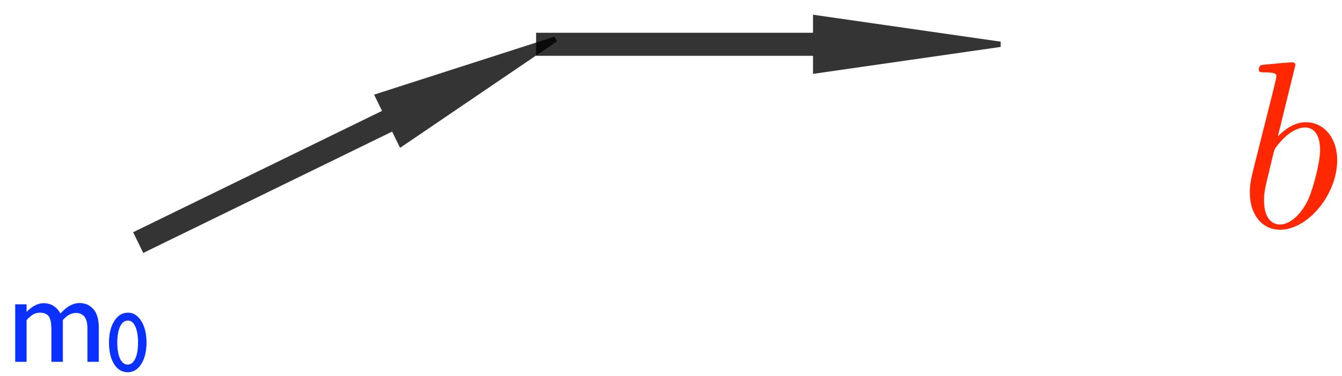

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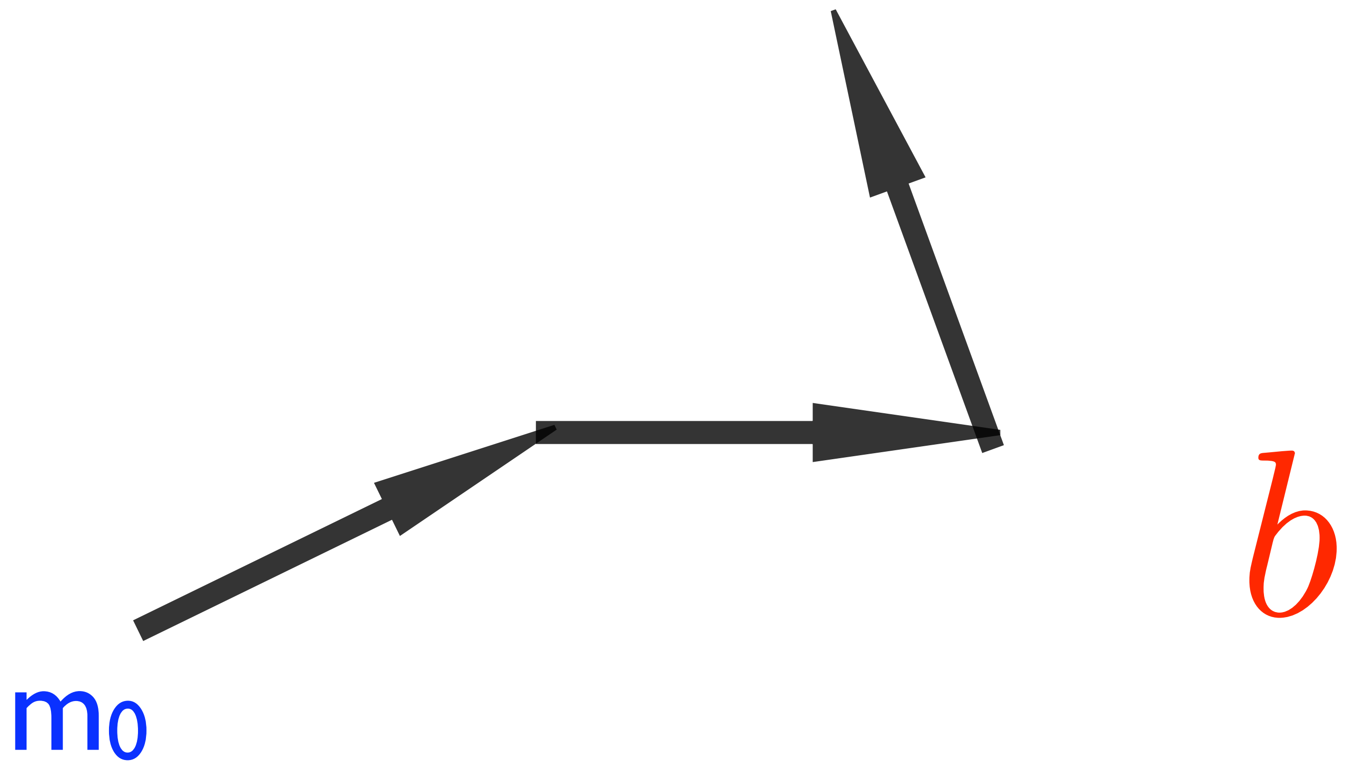

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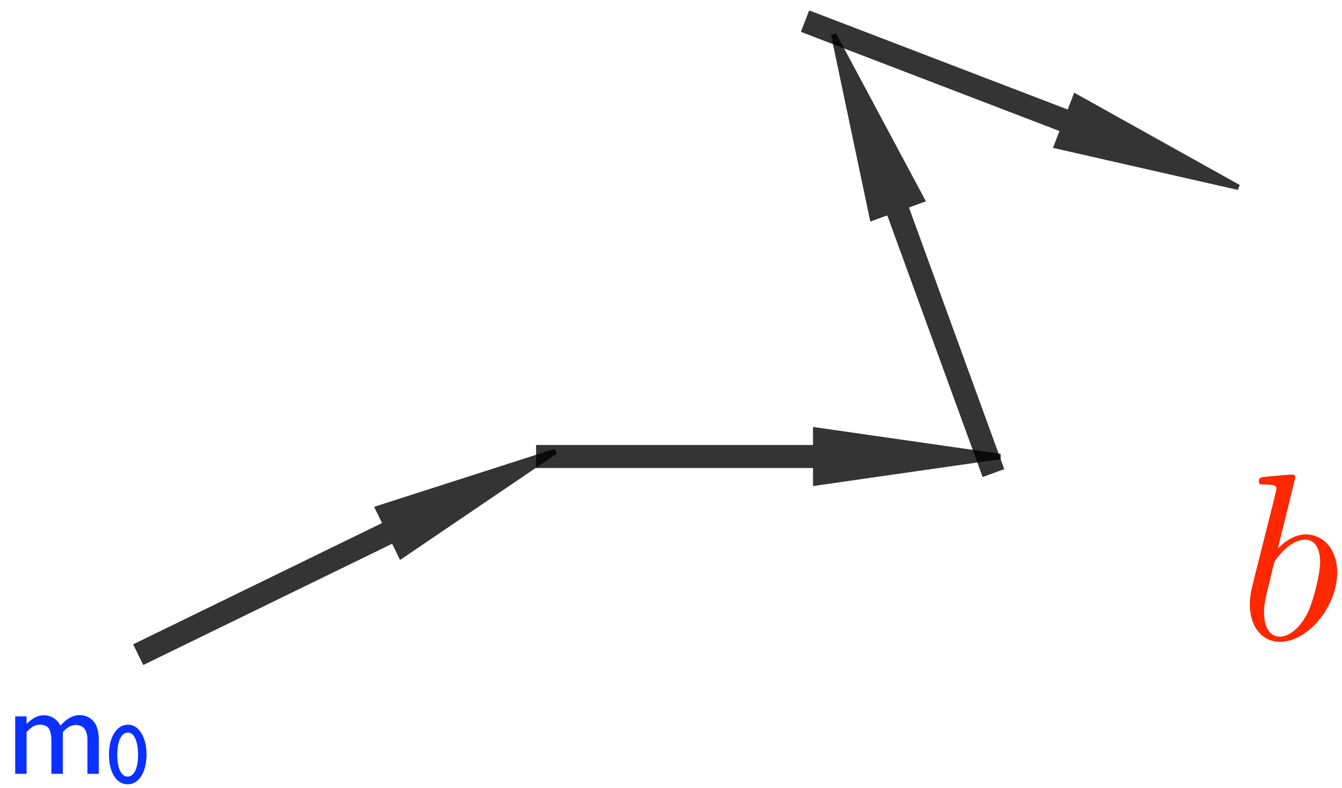

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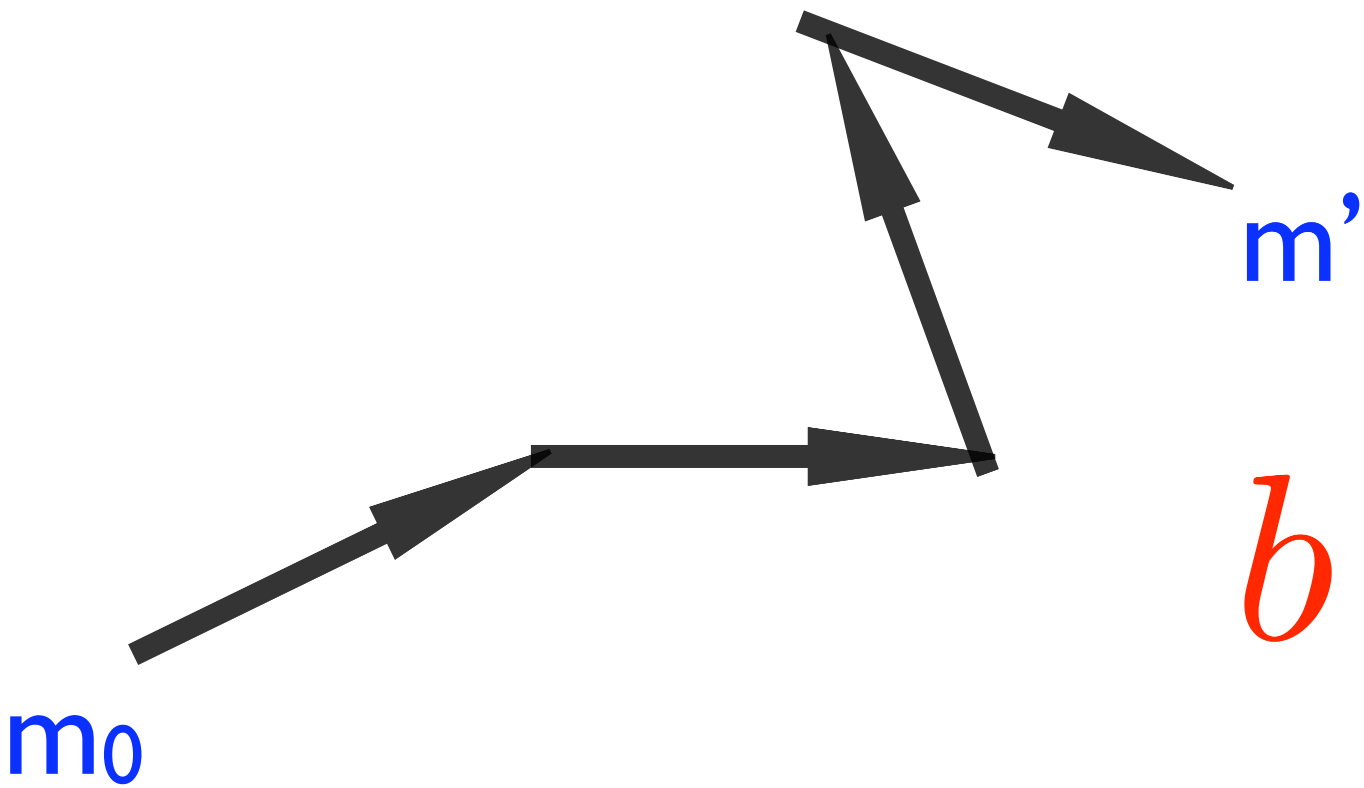

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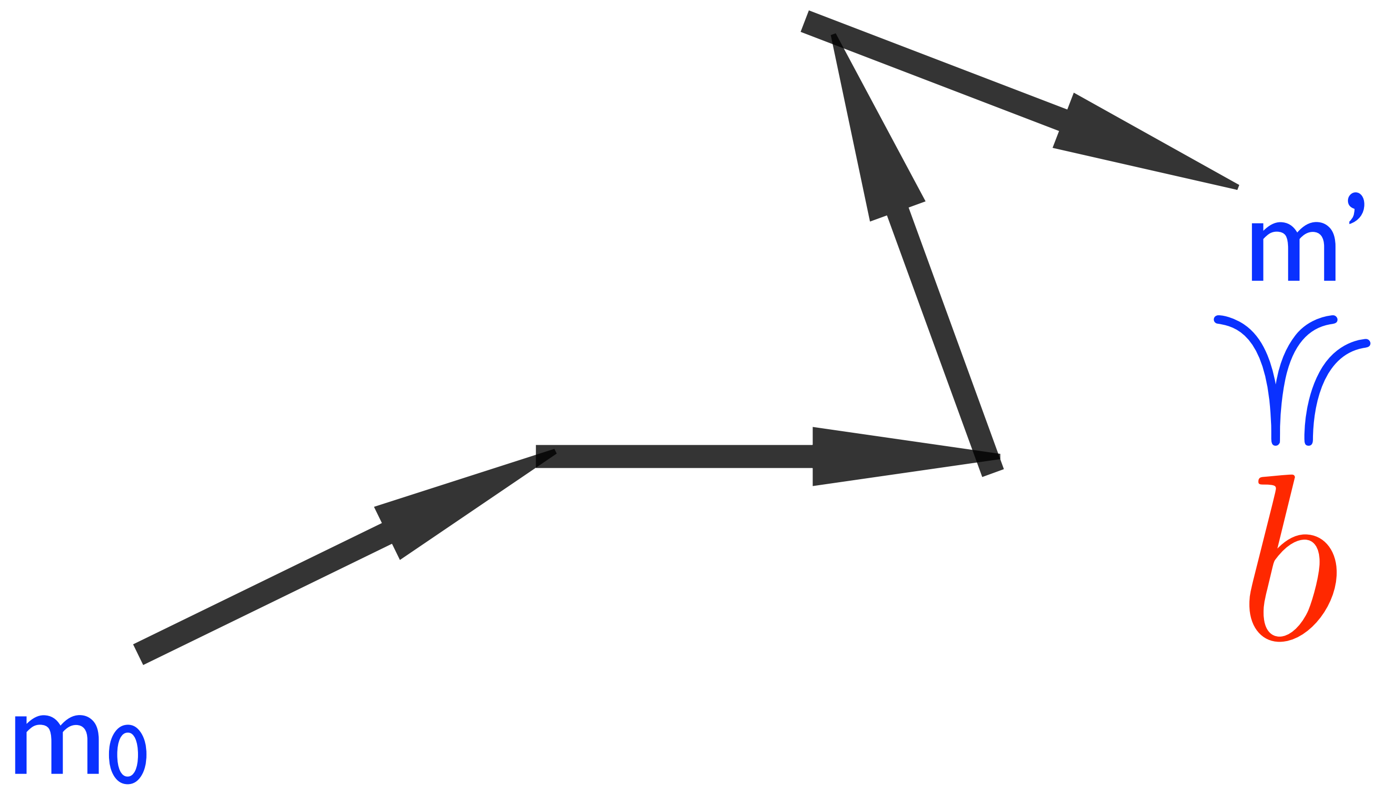

# The coverability problem

$b$

$m_0$

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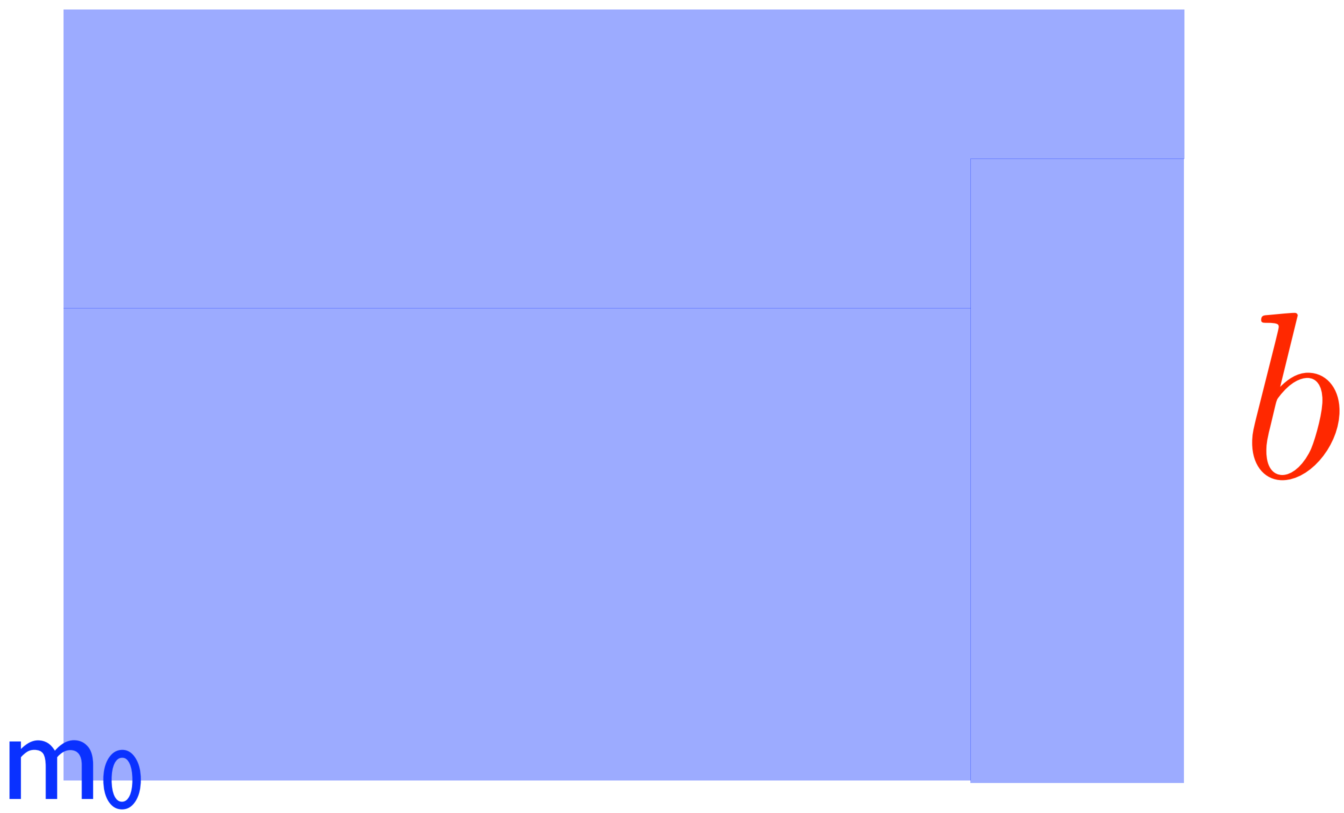

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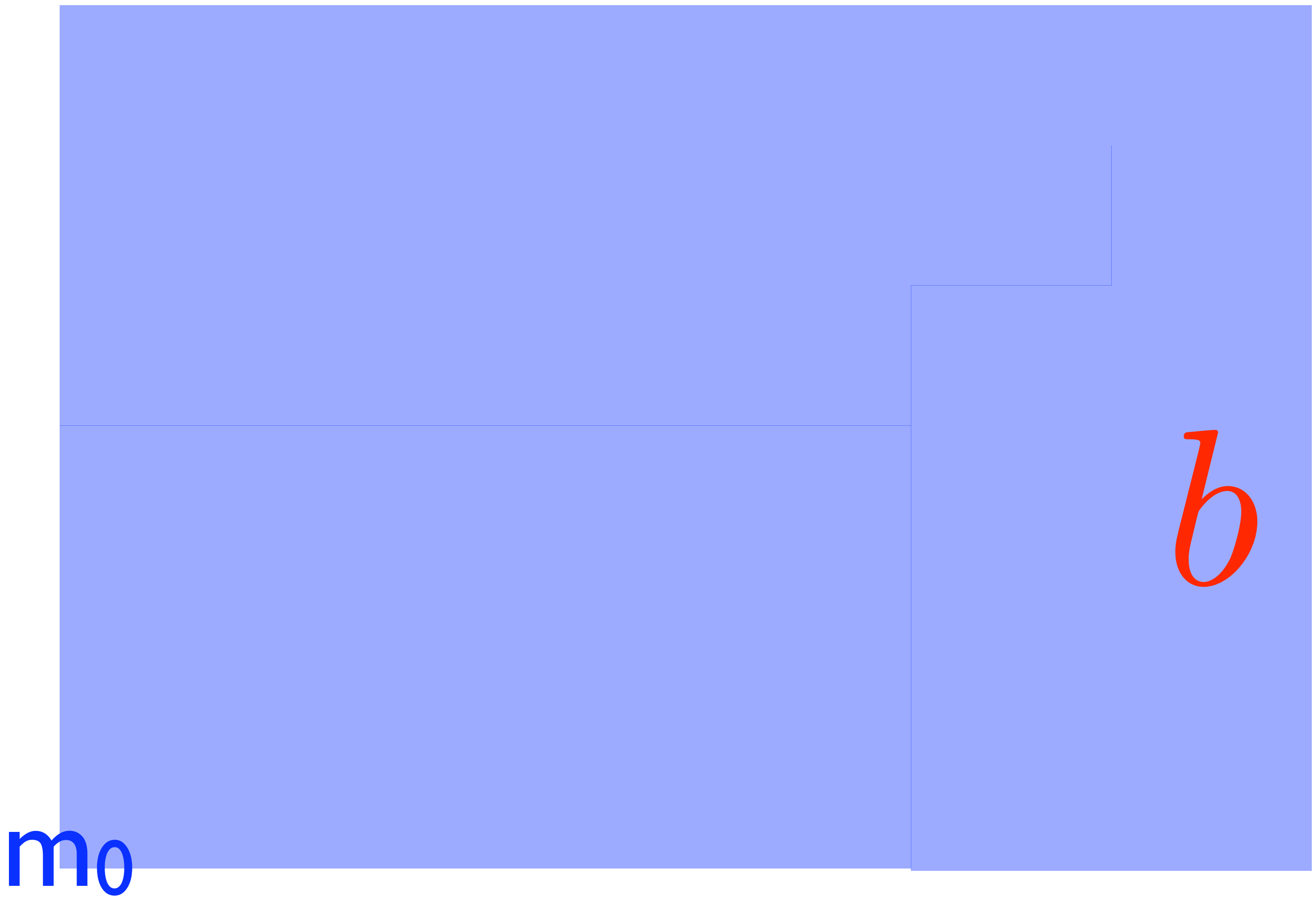

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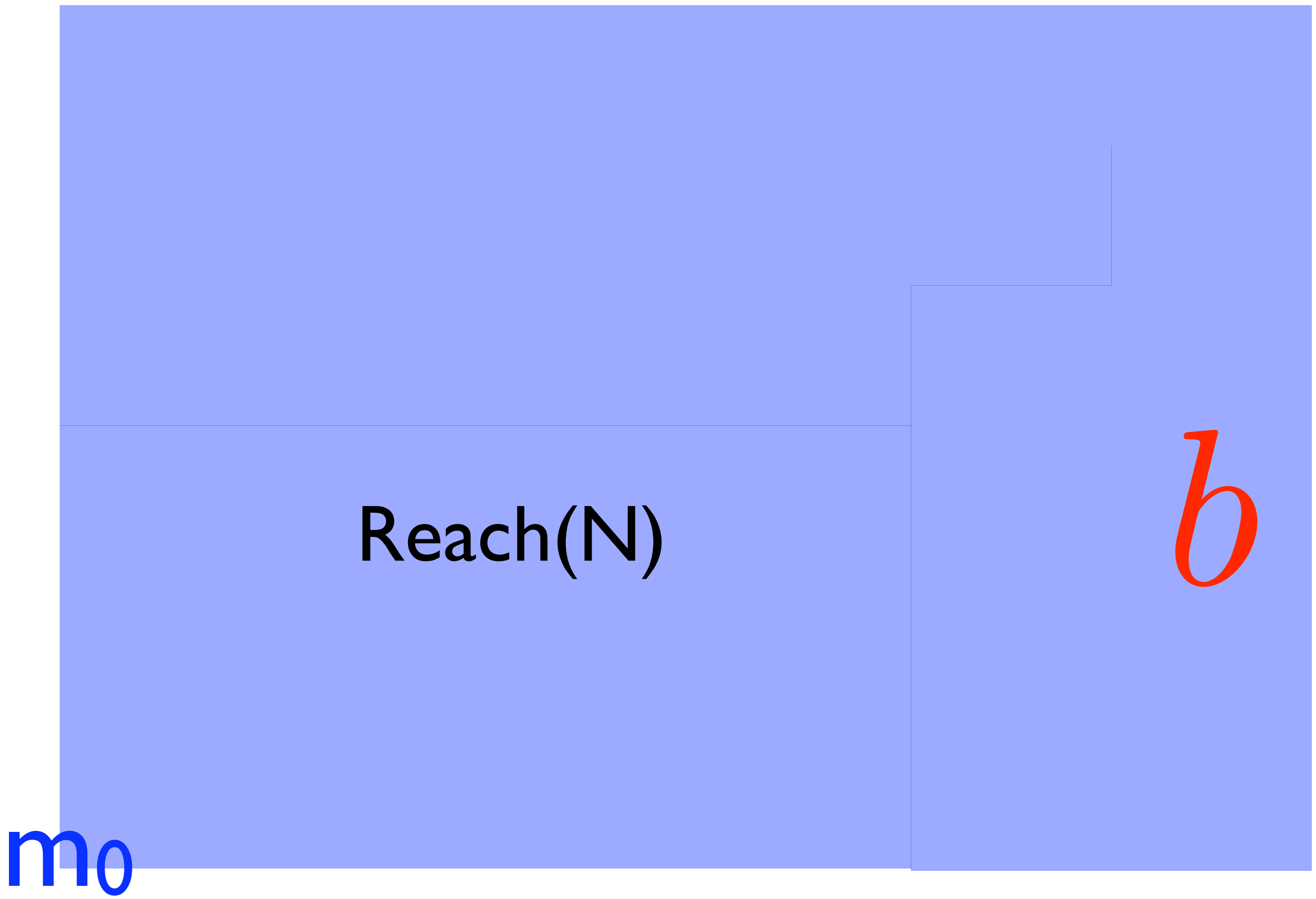

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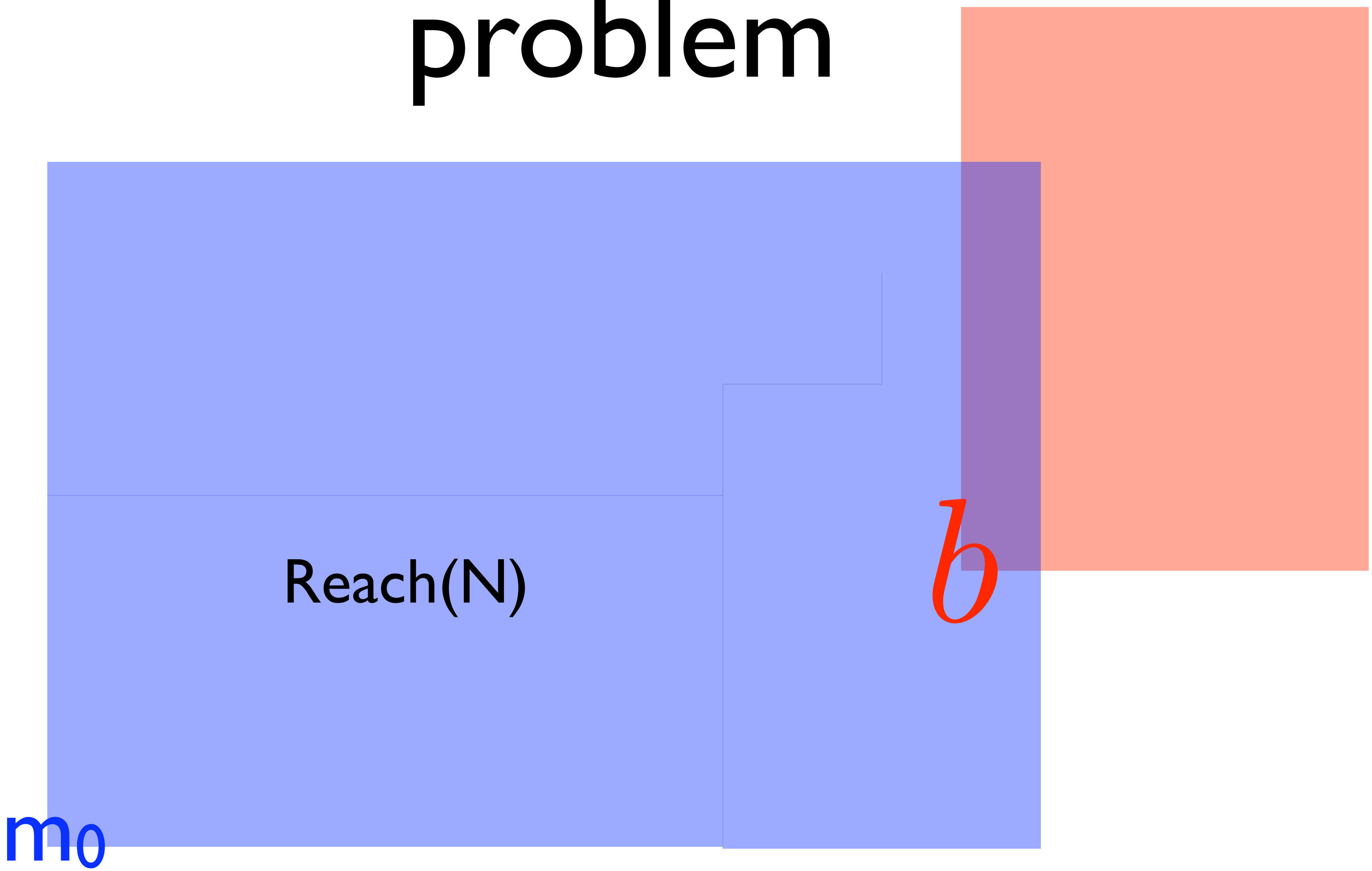

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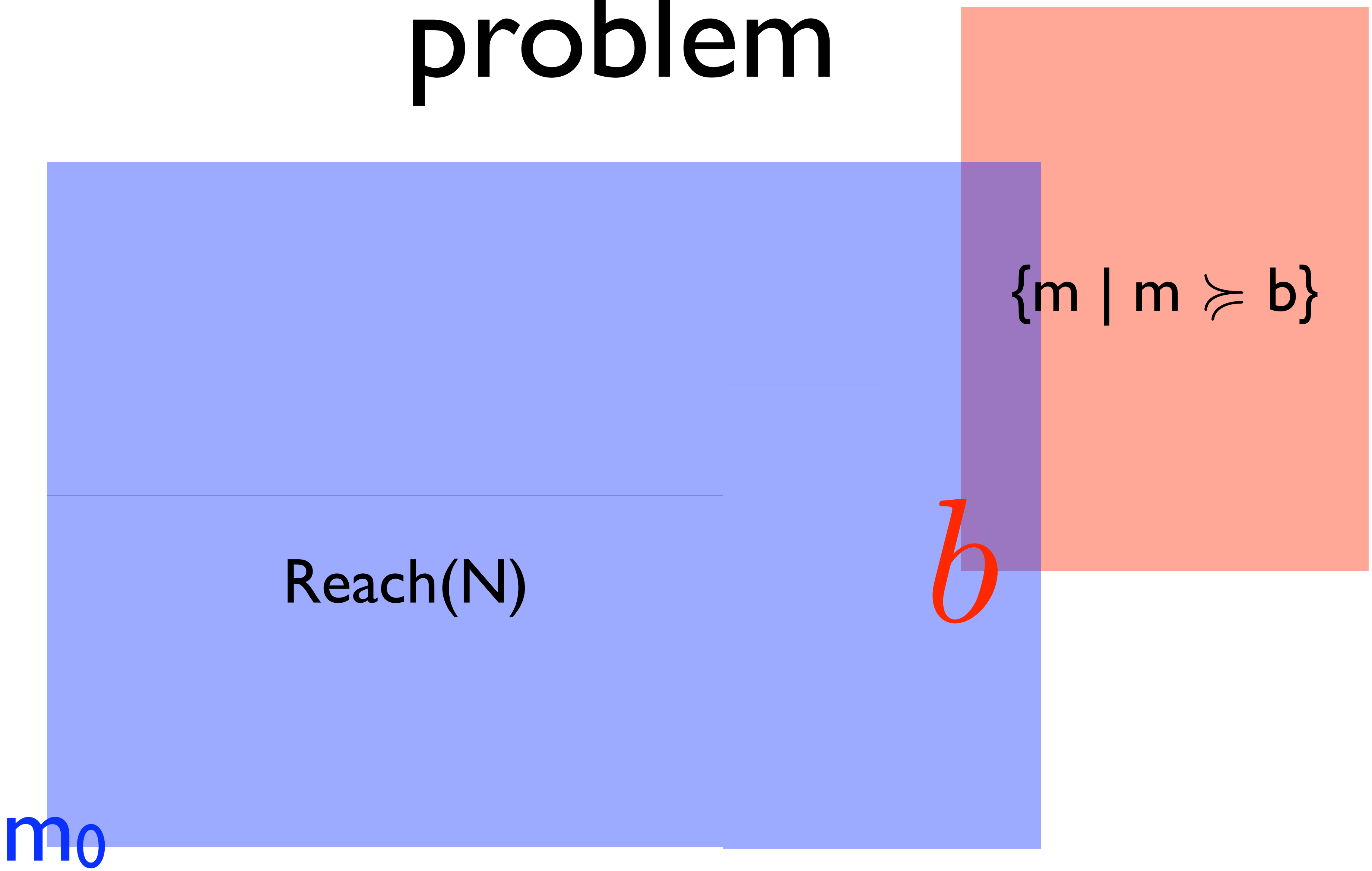

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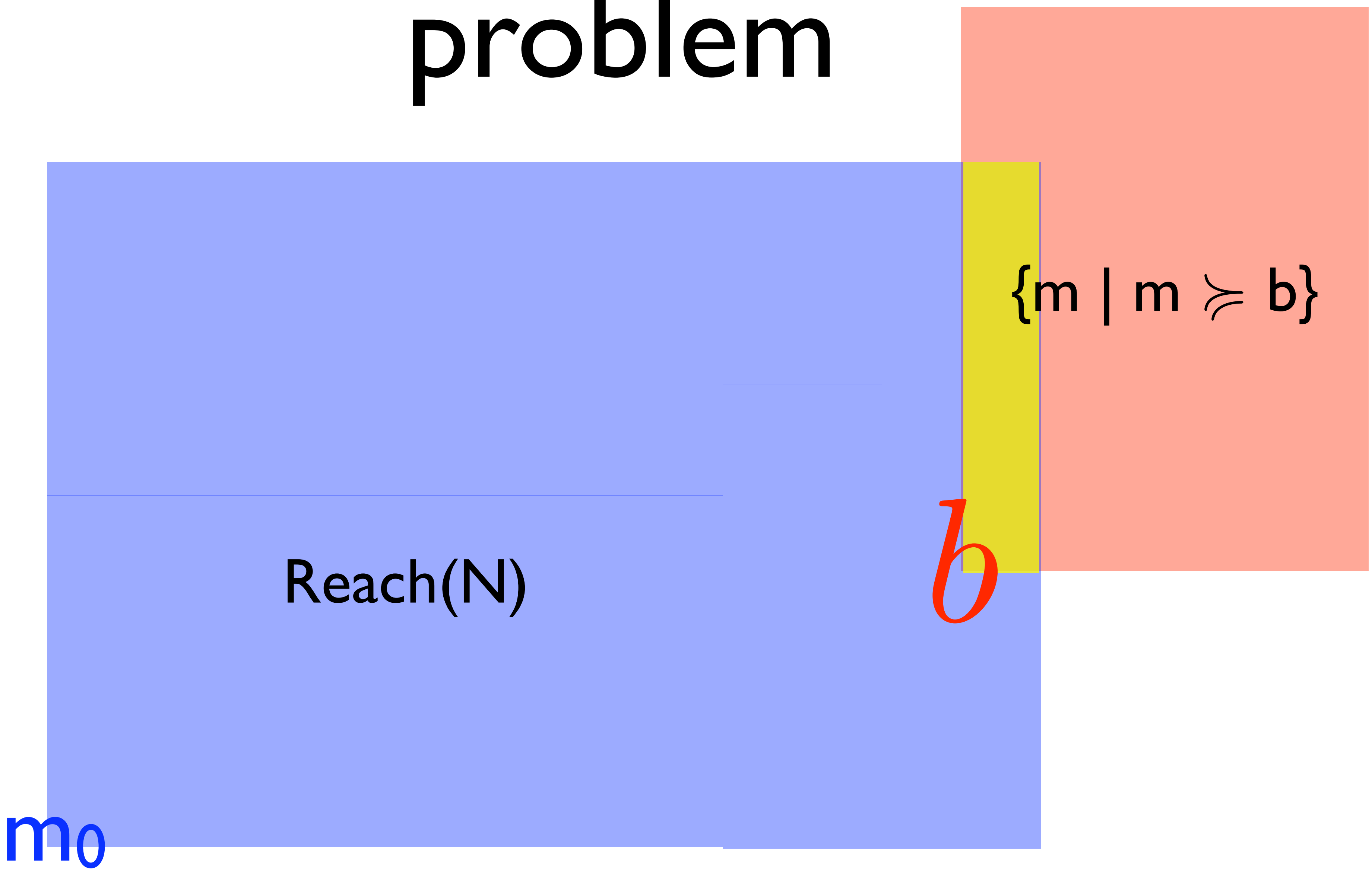


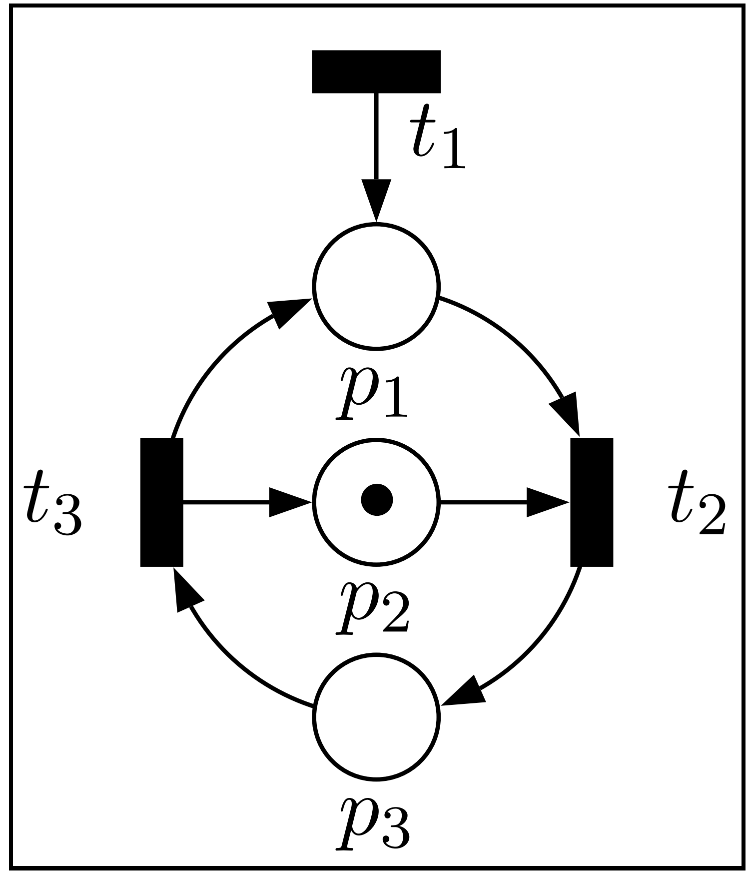
# The coverability problem

- Two alternative definitions:
  - Is there a reachable marking m s.t. $m \succcurlyeq b$ ?
  - Does $Reach(N) \cap \{m \mid m \succcurlyeq b\} \neq \phi$ ?

# Coverability: a natural question (indeed)

- Coverability might be regarded as the most natural reachability question in the framework of Petri nets
- Besides, coverability is much more easily solved than reachability

# Safety Properties



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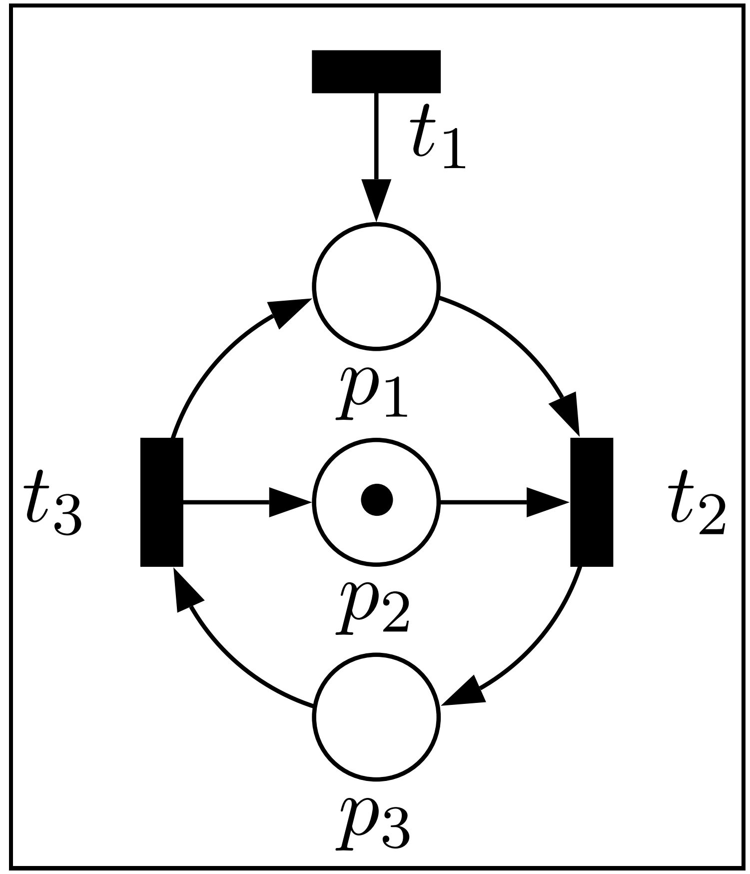
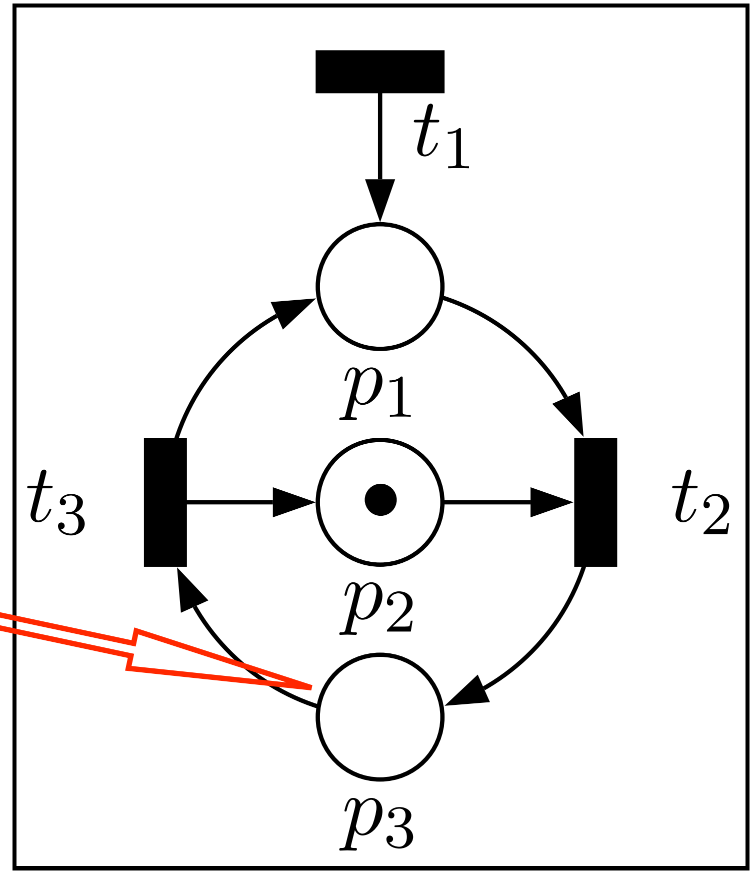



A marking m is unsafe when $m \succcurlyeq \langle 0, 0, 2, 0 \rangle$

# Safety Properties

No more than one token at a time in this place !!




A marking m is unsafe when $m \succcurlyeq \langle 0, 0, 2, 0 \rangle$

# First idea

- Use the coverability set !
- Remember: the coverability set over-approximates the reachable states:

$$\mathrm{Reach}(N) \subseteq \downarrow \mathrm{Cover}(N)$$

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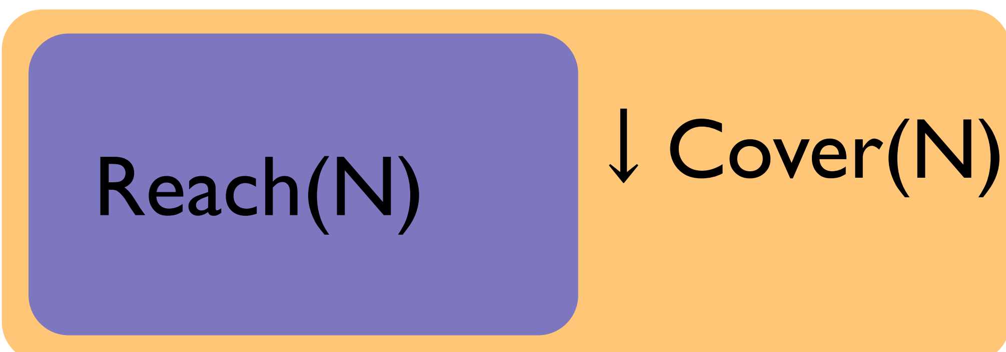

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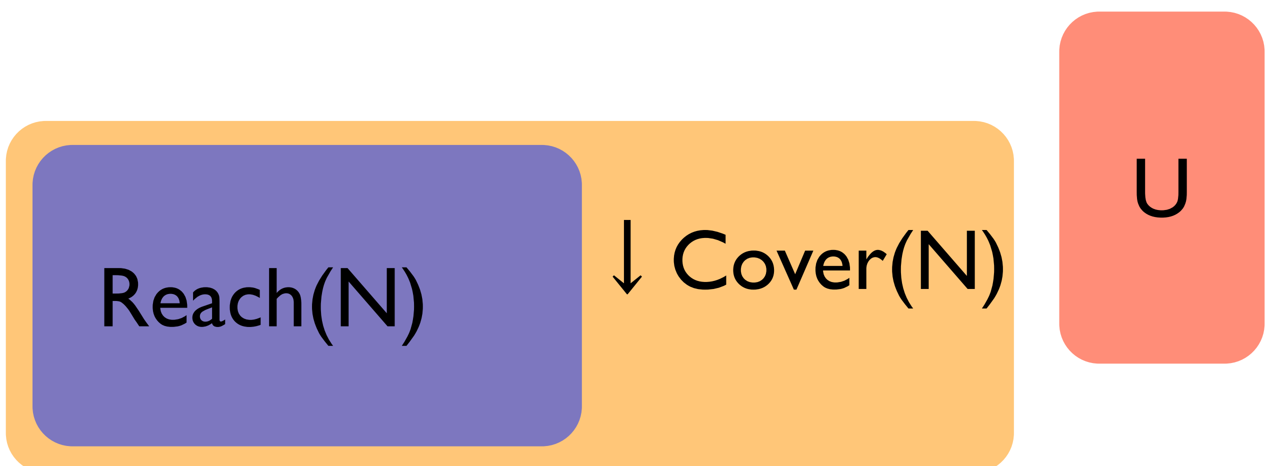

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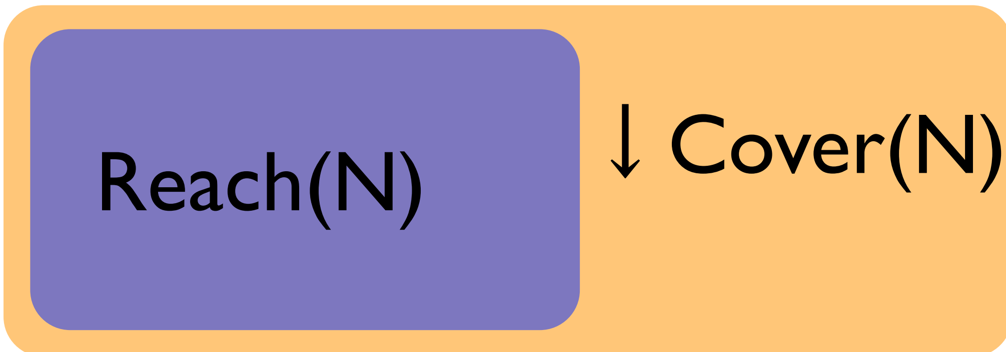
Reach(N)

# First idea



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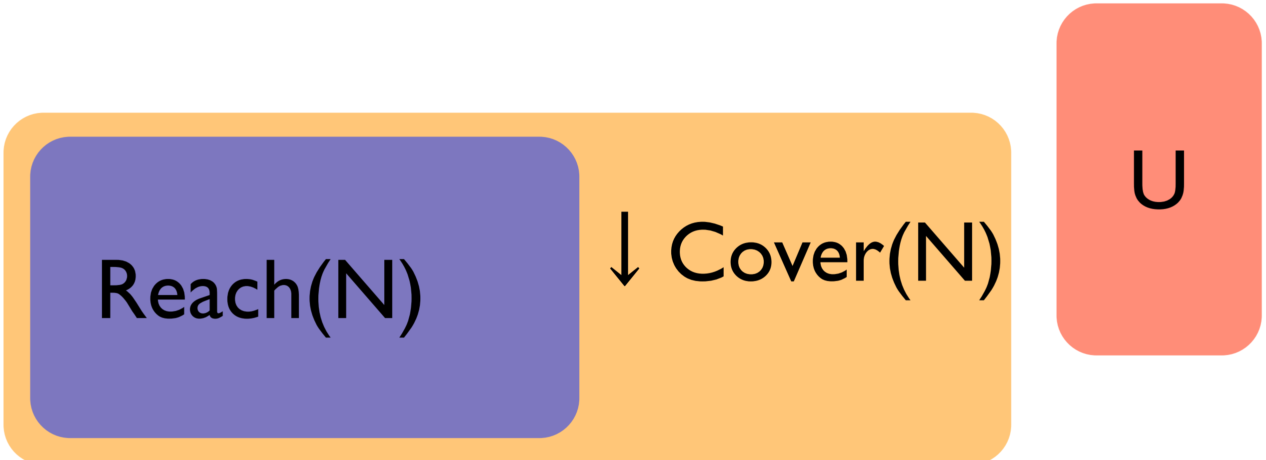

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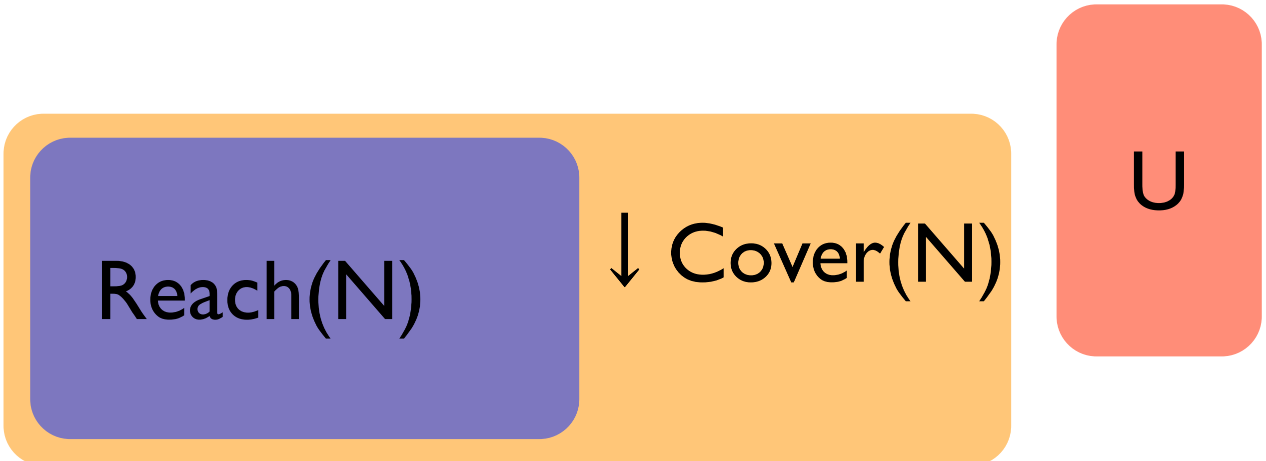
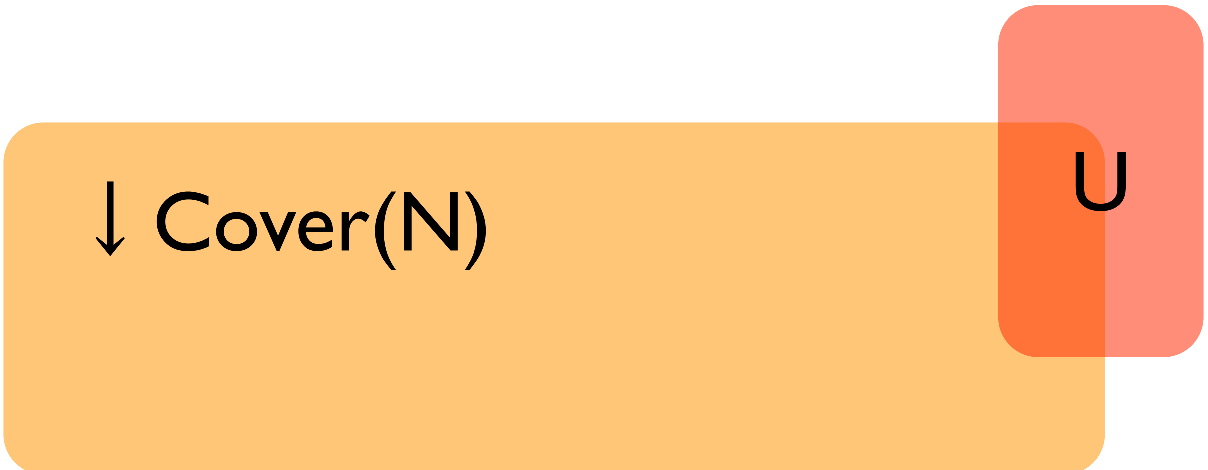



$$\downarrow \mathrm{Cover}(N) \cap U = \phi$$

implies

$$\mathrm{Reach}(N) \cap U = \phi$$

# What if ?




- There is m in $\downarrow$ Cover(N) $\cap$ U
- Hence, there is m' $\succcurlyeq$ m which is in Reach(N)
- However, any m' $\succcurlyeq$ m is also in U
- Thus, there is m' both in Reach(N) and U

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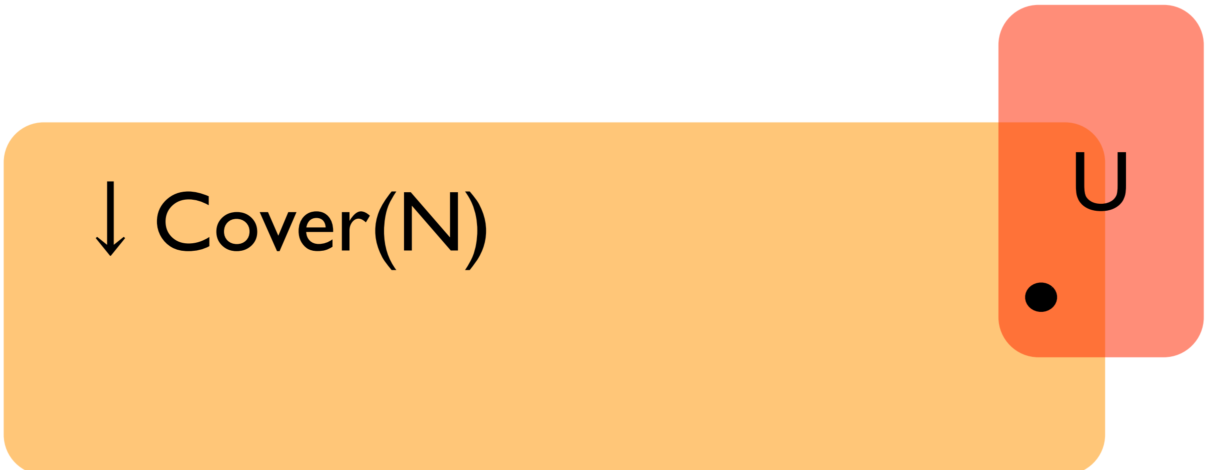


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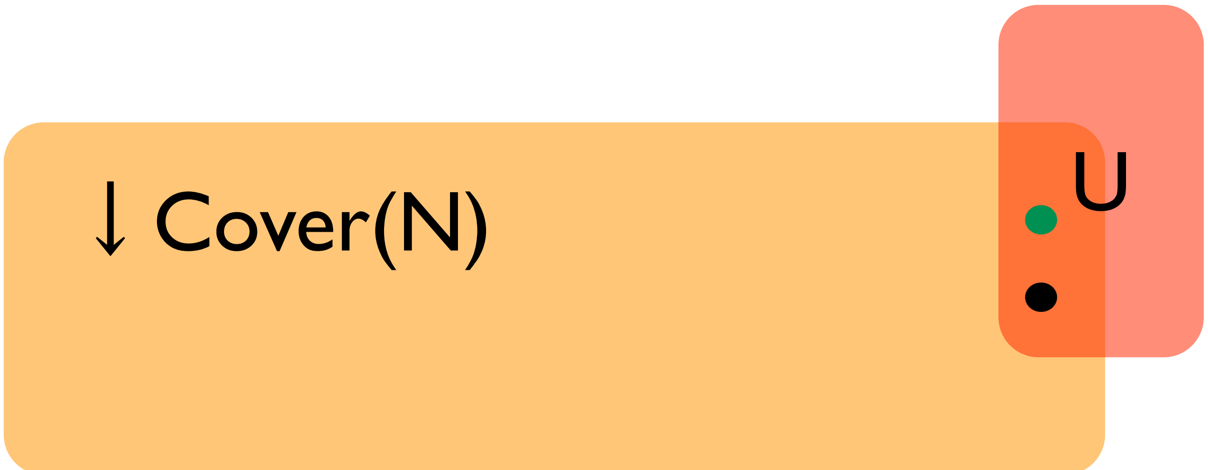


- There is m in ↓ Cover(N) ∩ U
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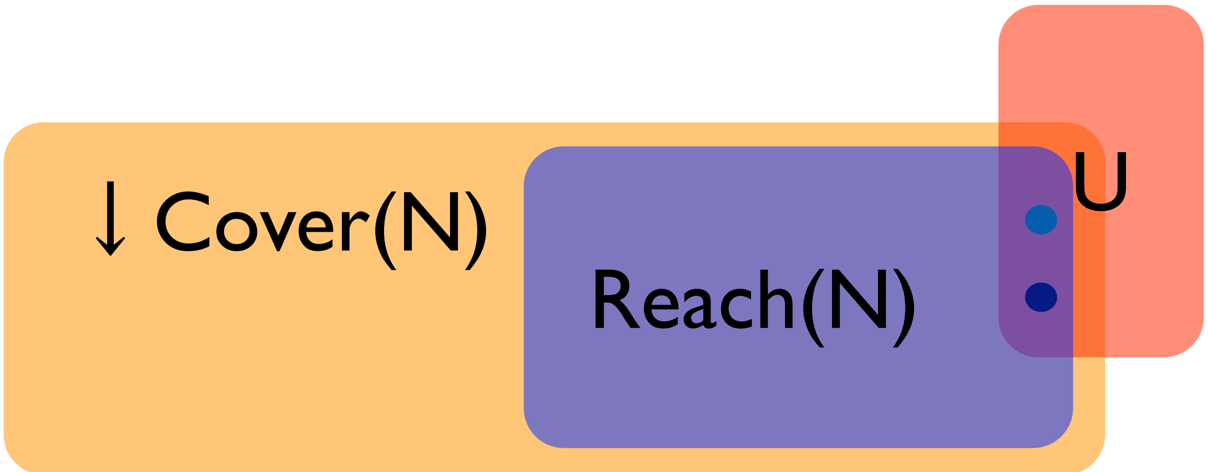


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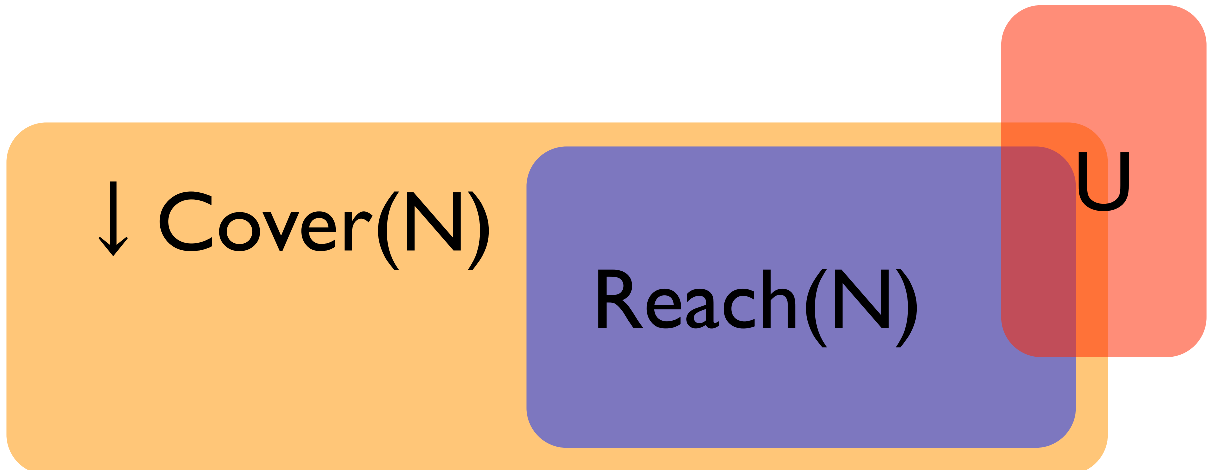

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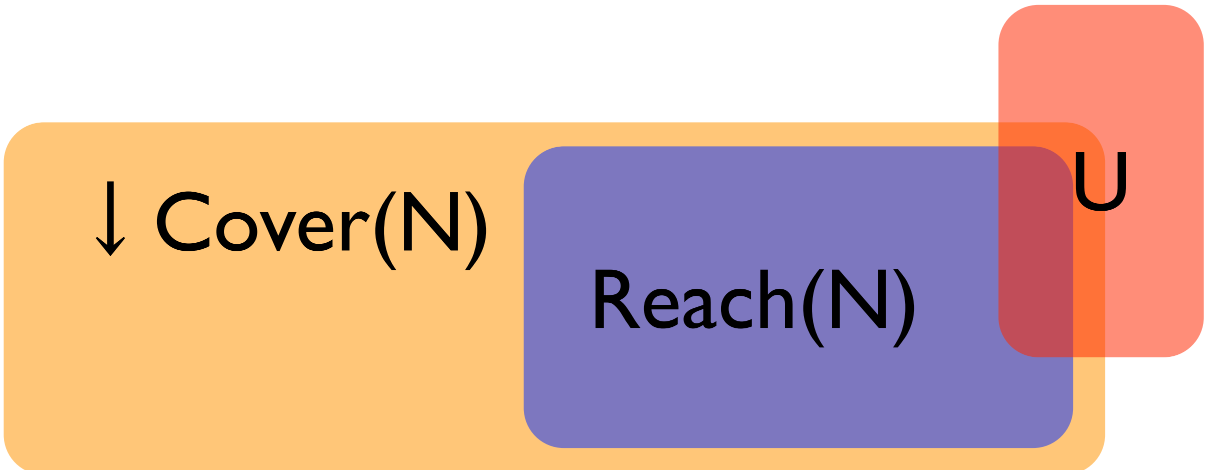


$$\text{Reach}(N) \cap U = \emptyset$$

implies

$$\downarrow \text{Cover}(N) \cap U = \emptyset$$

# Coverability set and coverability problem

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- Theorem:

  $\mathrm{Reach}(N) \cap U = \phi$ iff $\downarrow \mathrm{Cover}(N) \cap U = \phi$

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- ...but $U$ and $\downarrow Cover(N)$ might both be infinite !

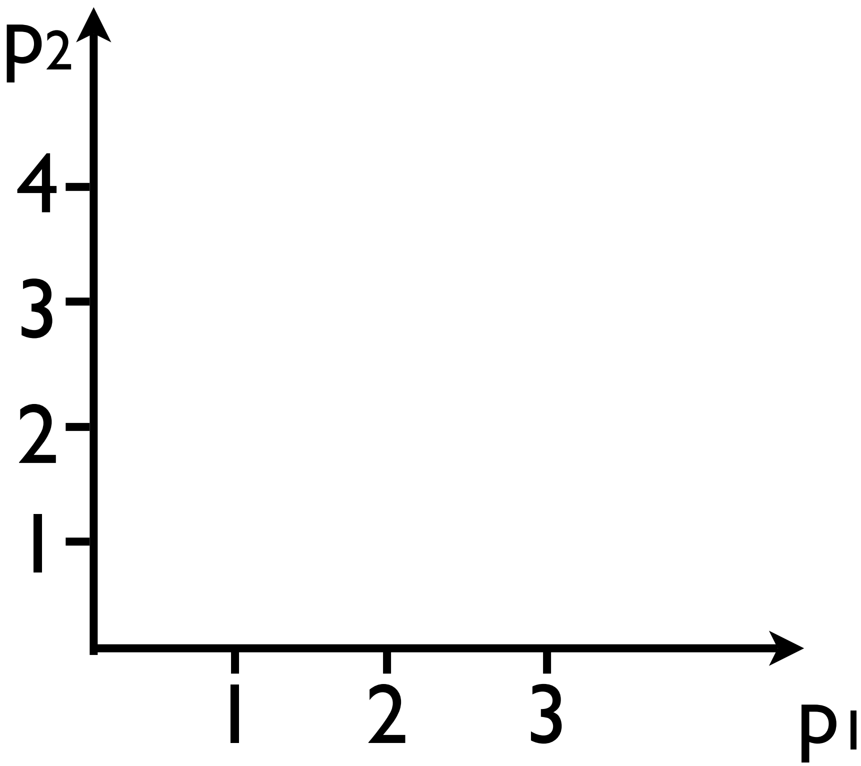
# Coverability set and coverability problem

- Theorem:

  $\mathrm{Reach}(N) \cap U = \emptyset$ iff $\downarrow \mathrm{Cover}(N) \cap U = \emptyset$

- Nice,...
- ...but $U$ and $\downarrow \mathrm{Cover}(N)$ might both be infinite !
- How do we test that $\downarrow \mathrm{Cover}(N) \cap U = \emptyset$??

# Coverability set and coverability problem



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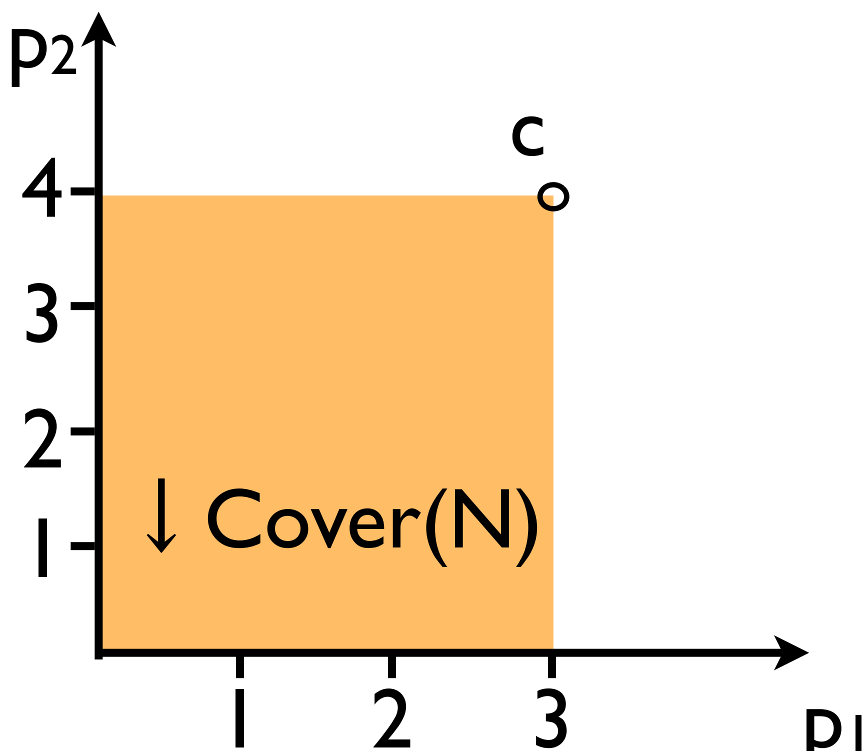

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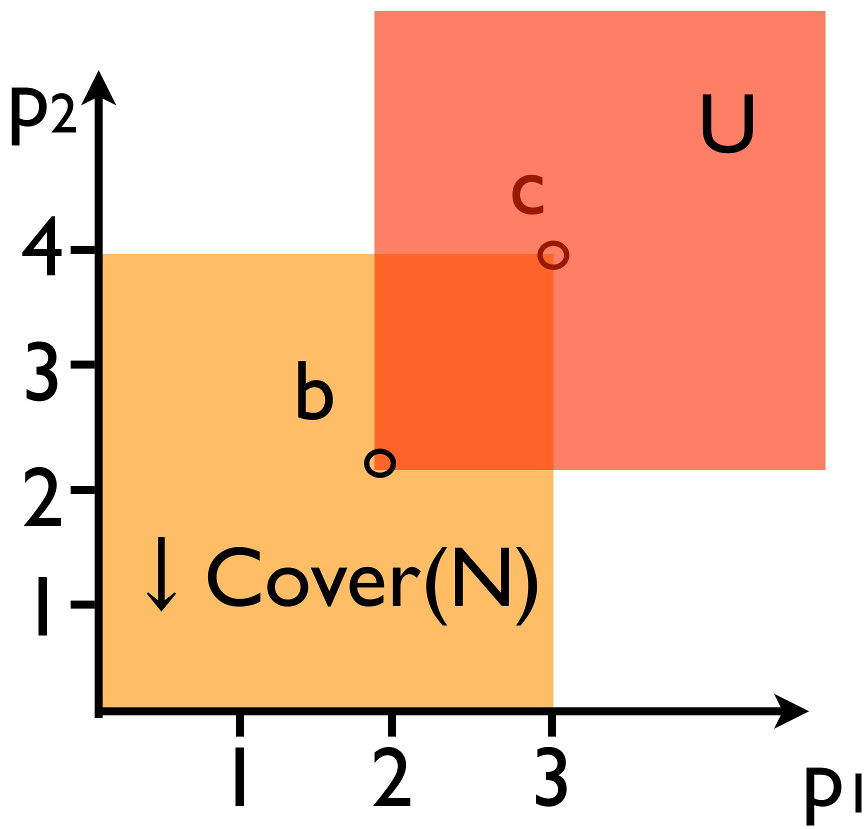
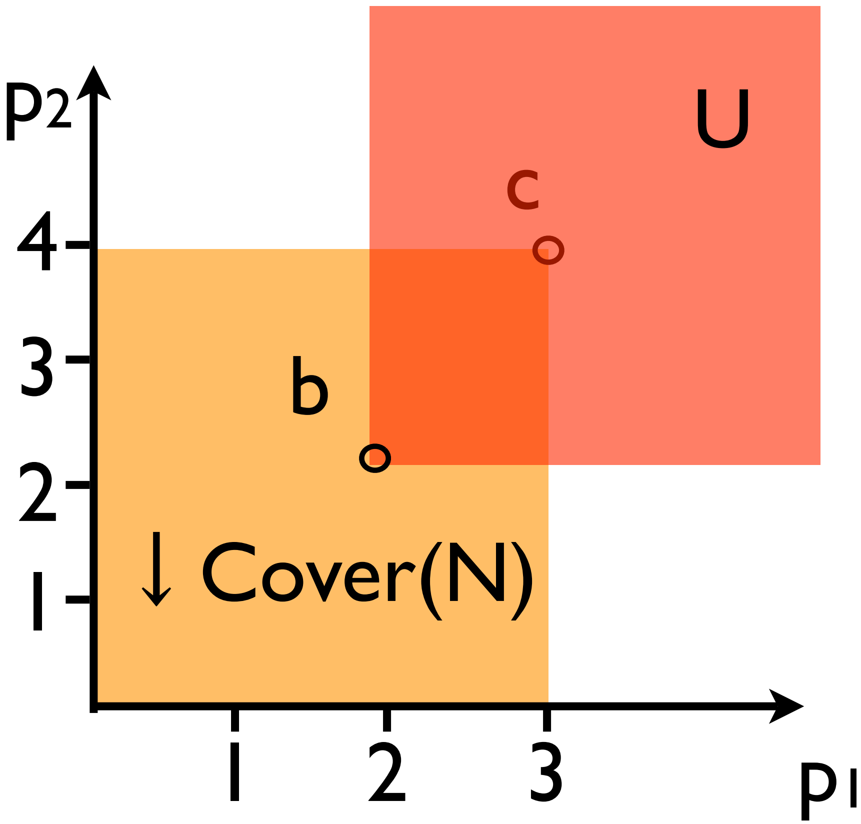


# Coverability set and coverability problem

$c \not\succcurlyeq b$



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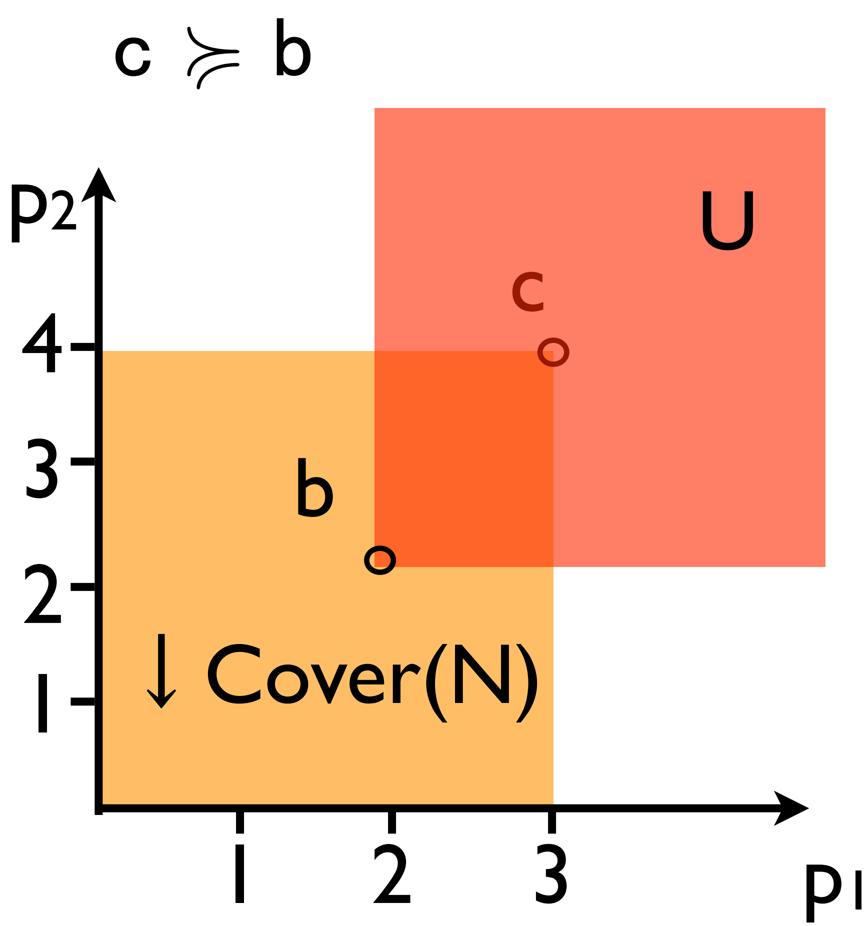
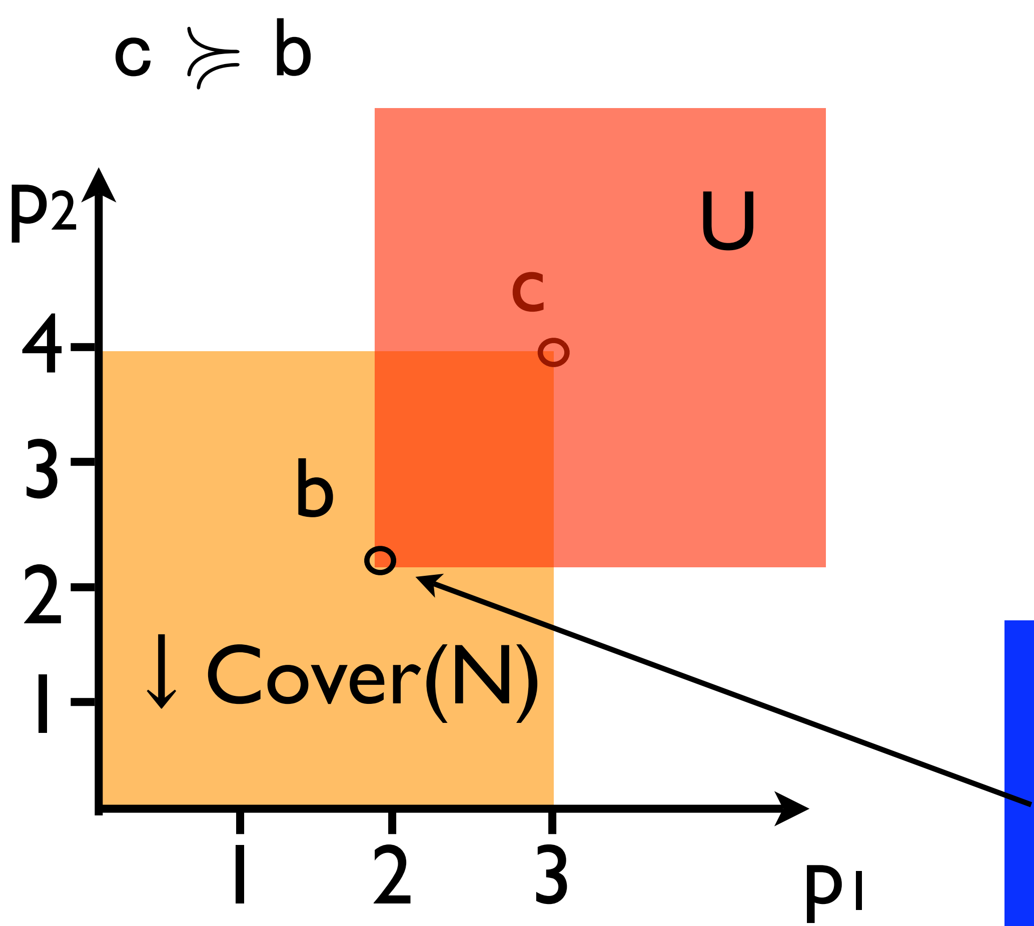



All we need to remember is the (finite) set of minimal elements Min(U)

# Coverability set and coverability problem

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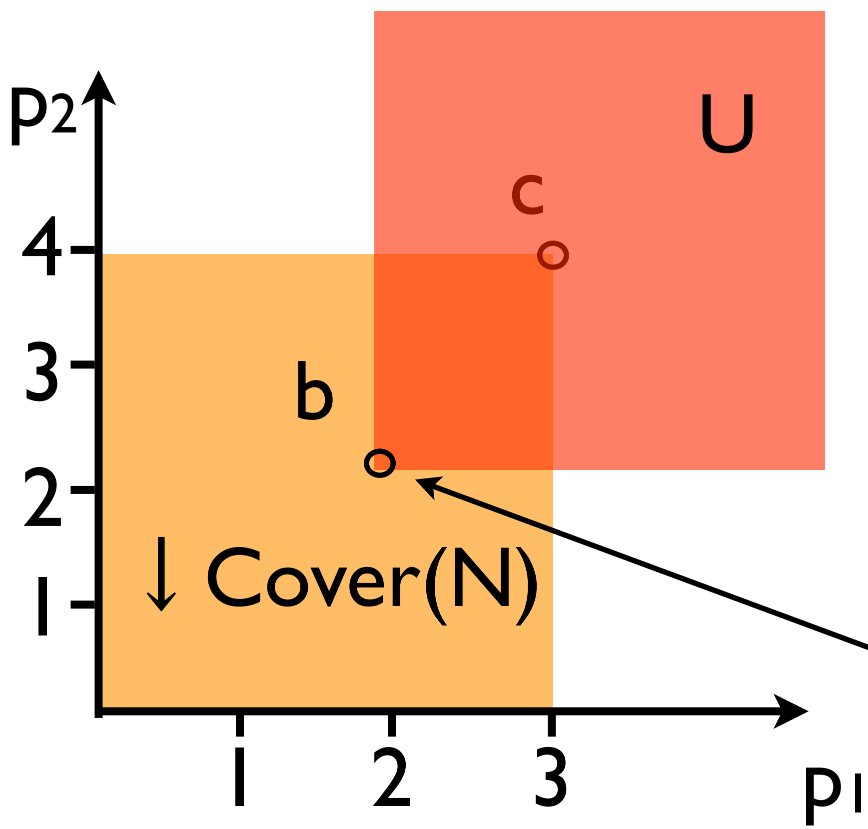
# Coverability set and coverability problem

$c \succcurlyeq b$



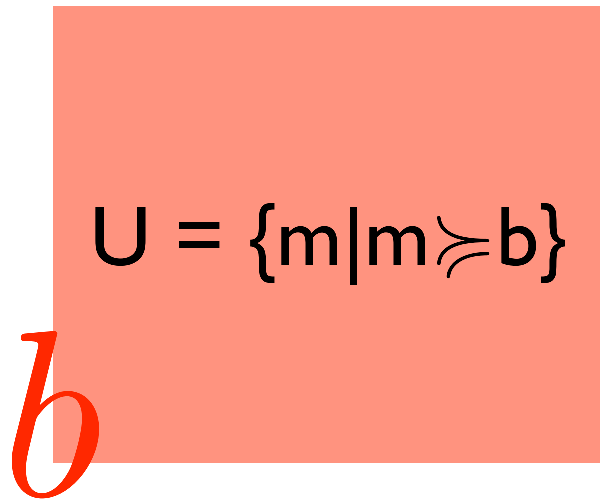

$\downarrow$ Cover(N) $\cap$ U $\neq \phi$

iff

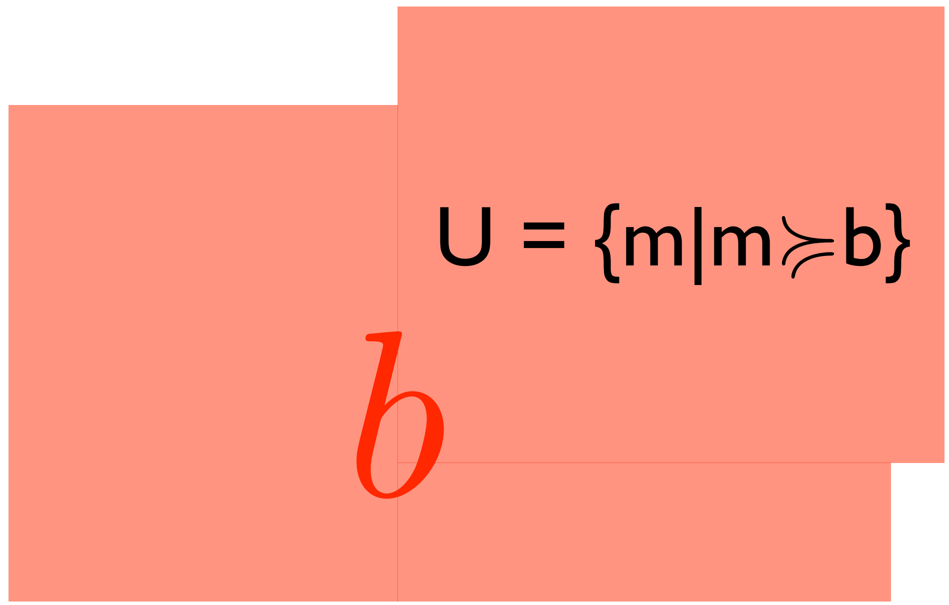
there is c in Cover(N) and b in Min(U) s.t. $c \succcurlyeq b$

All we need to remember is the (finite) set of minimal elements Min(U)

# Backward approach

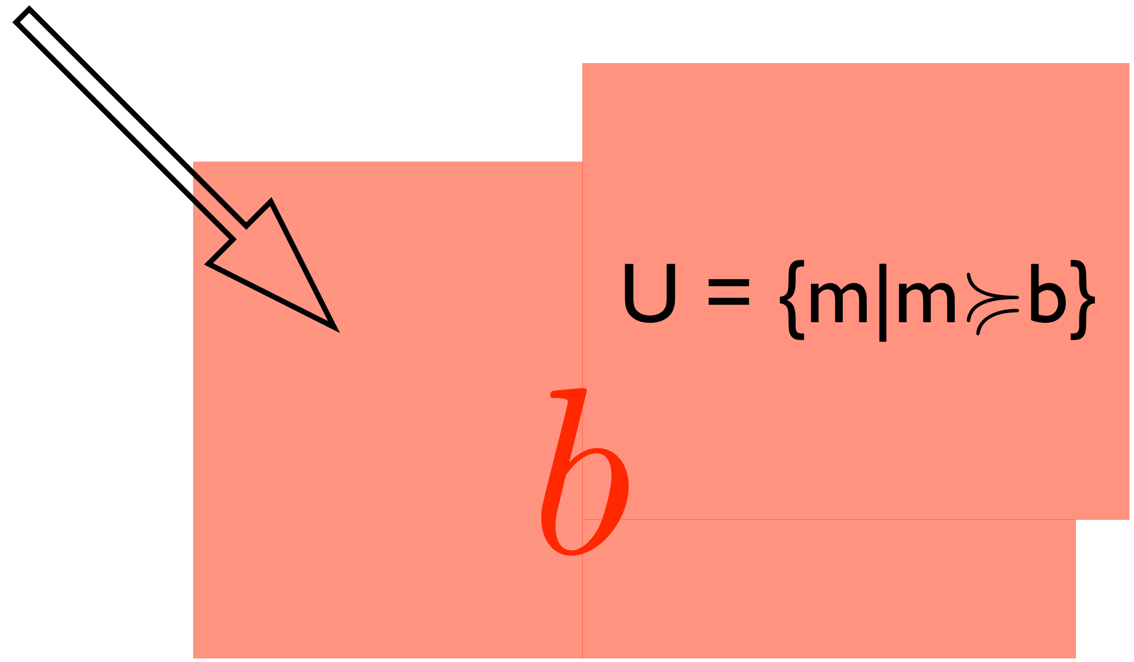


# Backward approach



# Backward approach

All the markings that can reach U in one step



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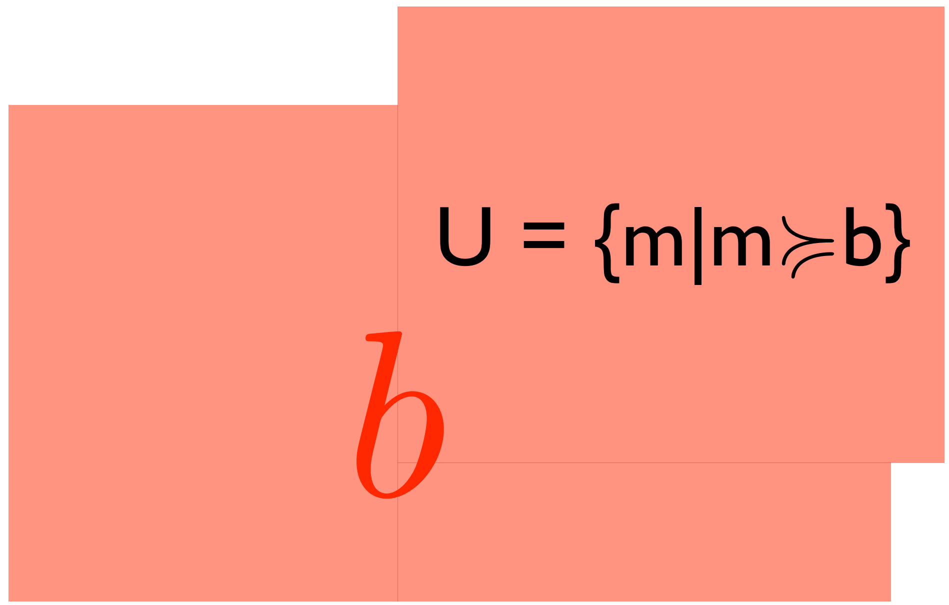

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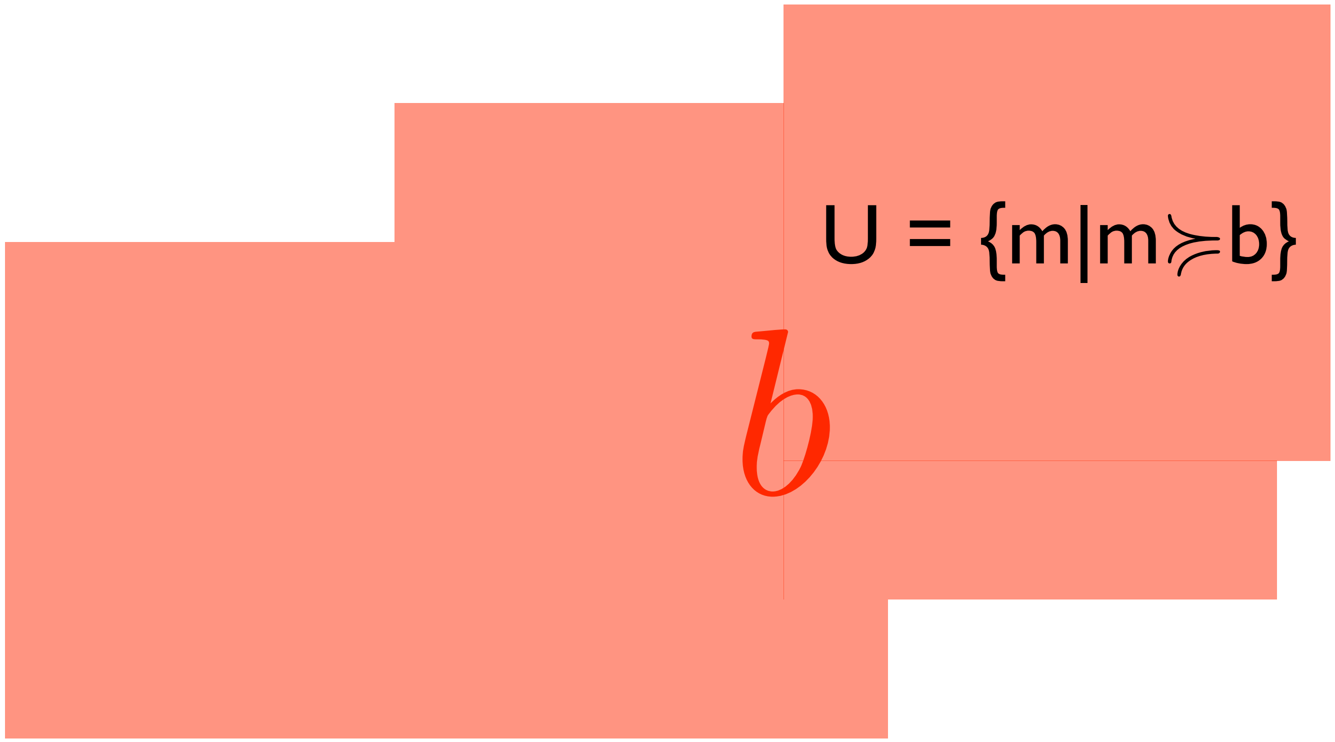

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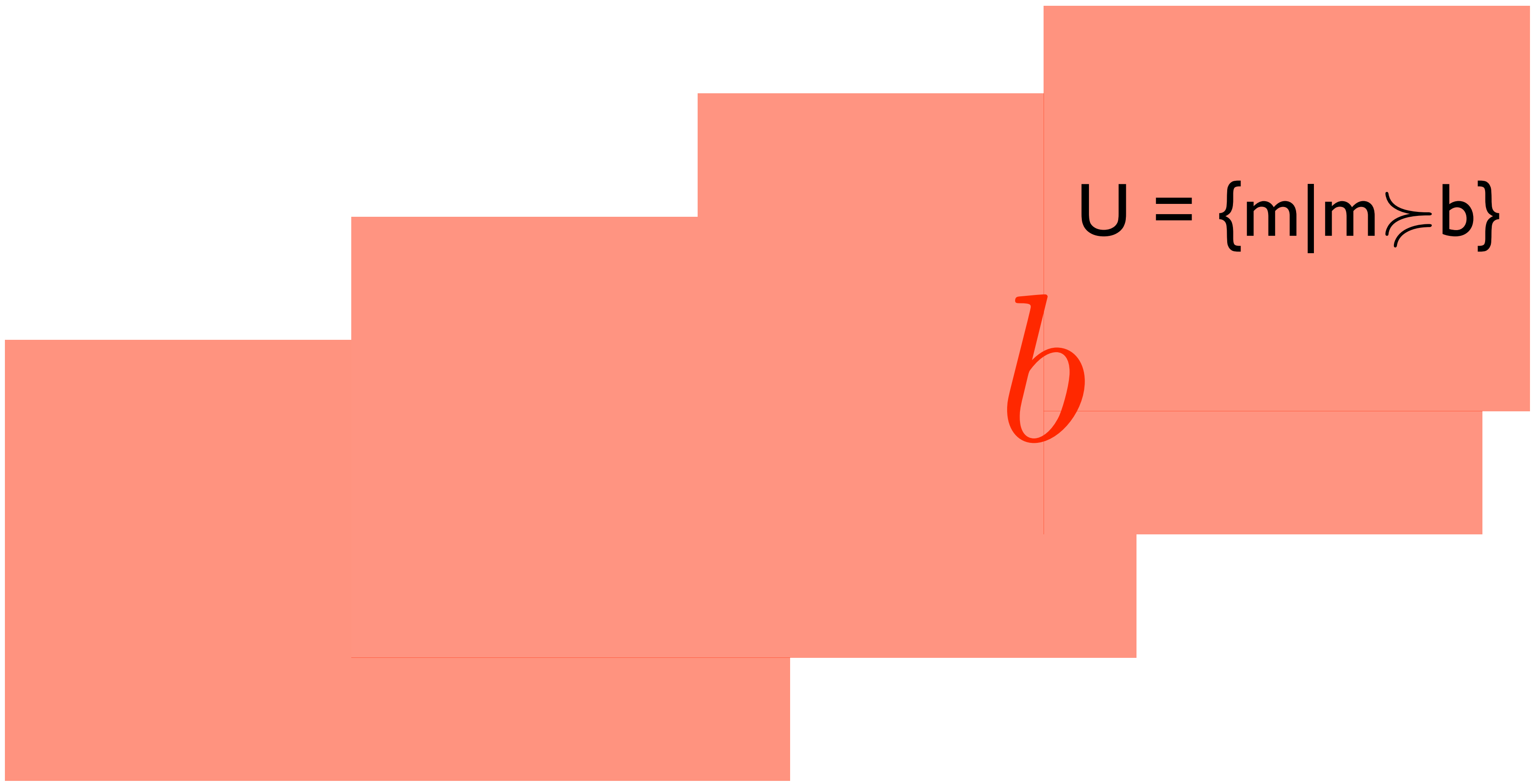
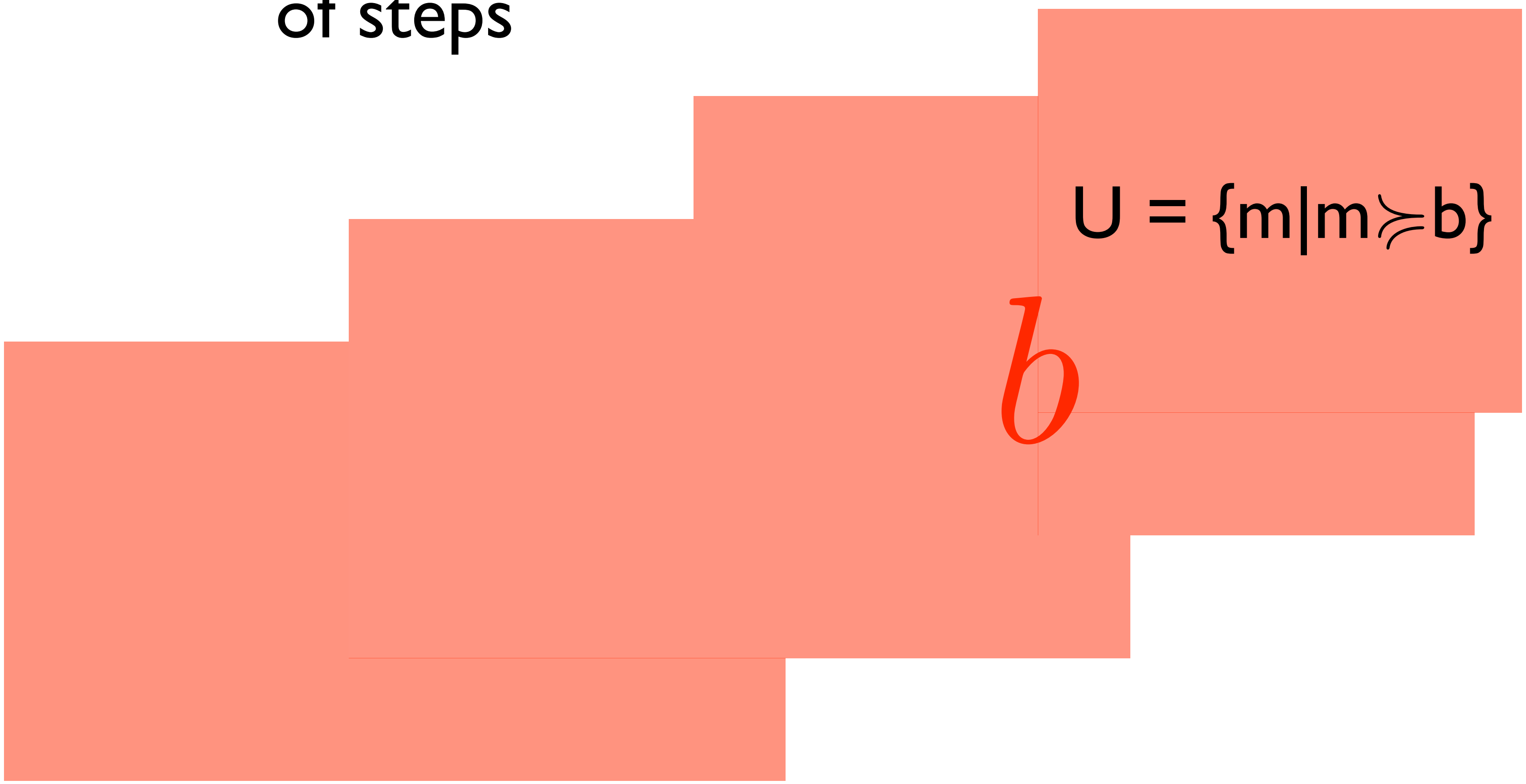


# Backward approach

In the end, we want to obtain all the markings that can reach U in any number of steps



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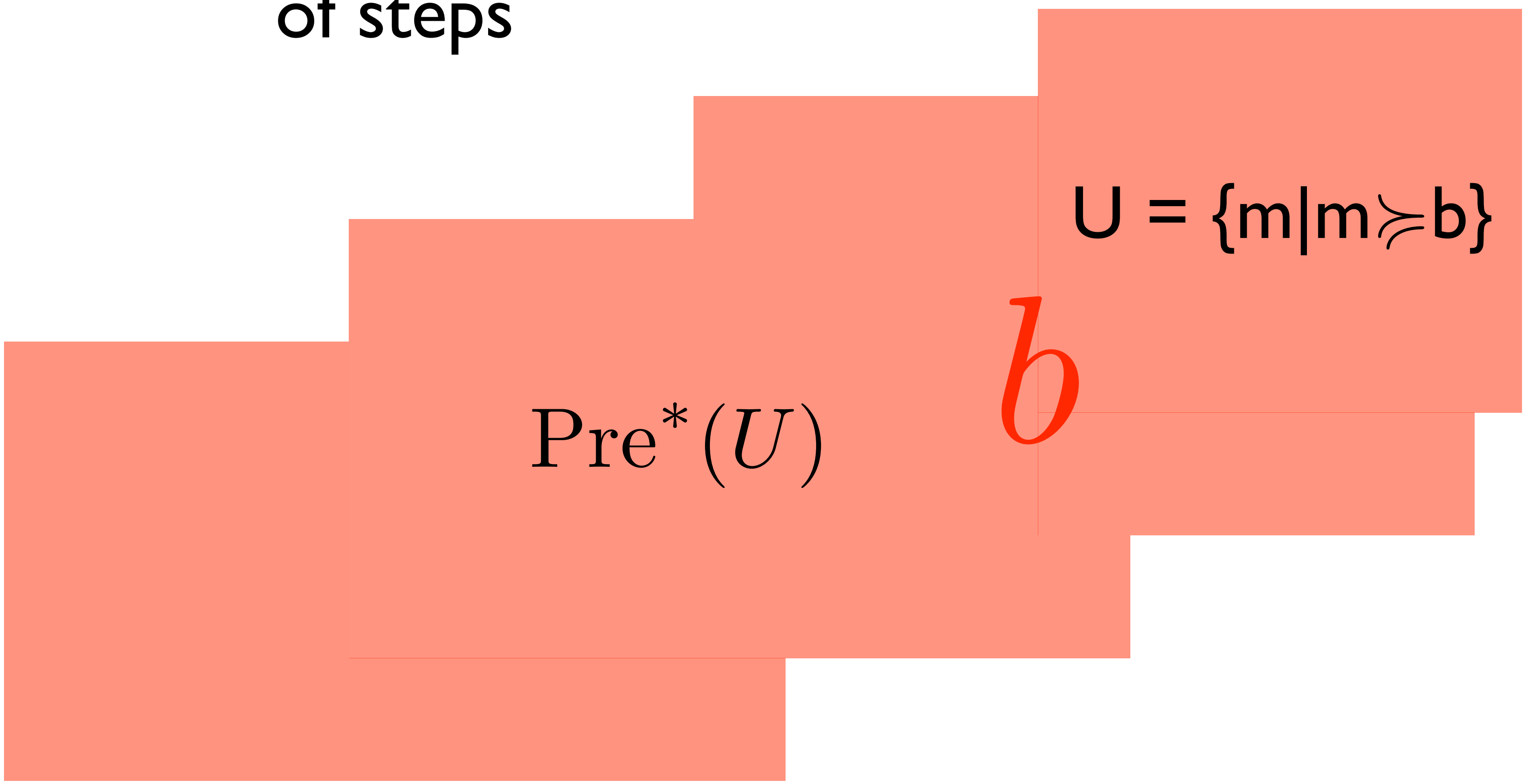

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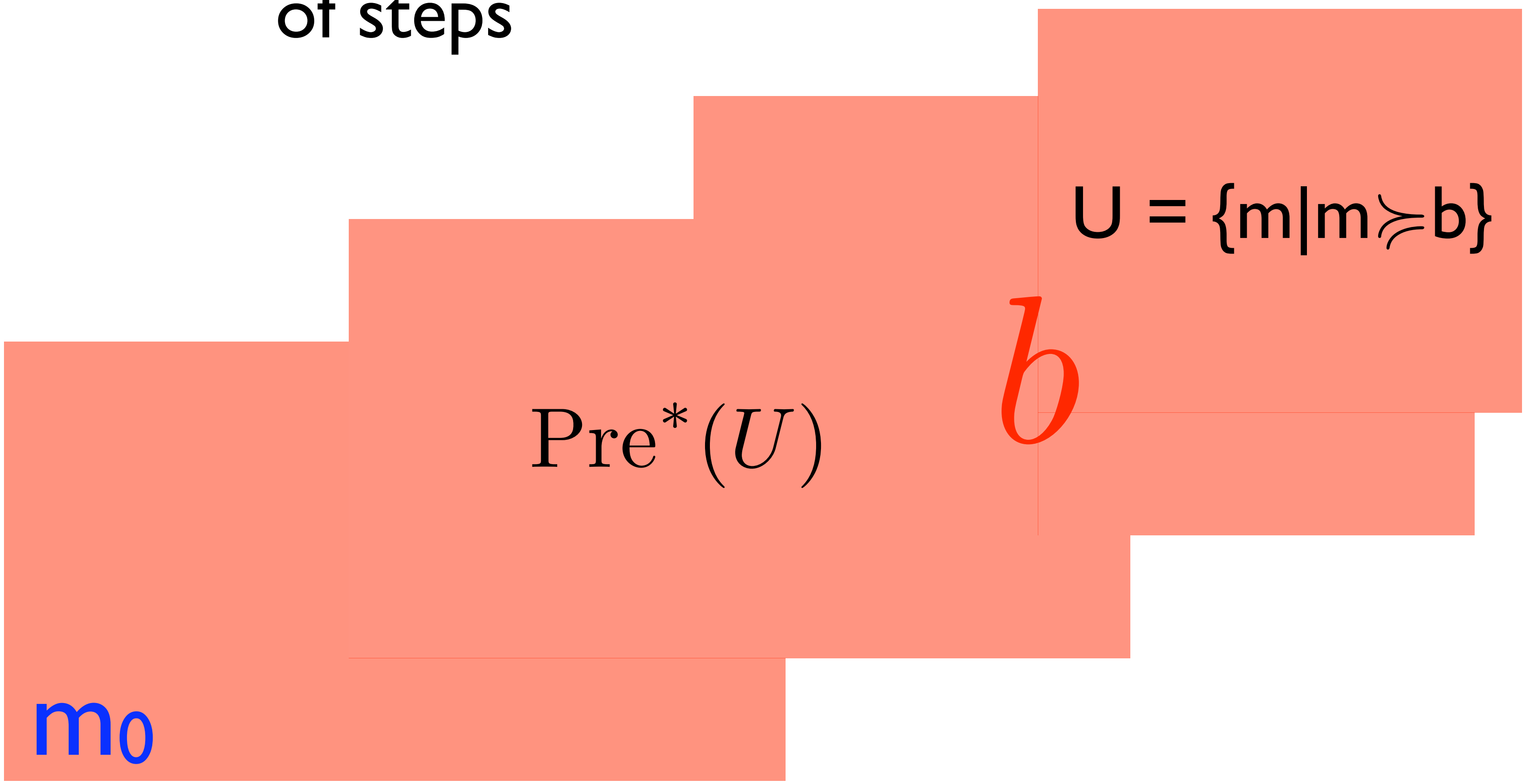

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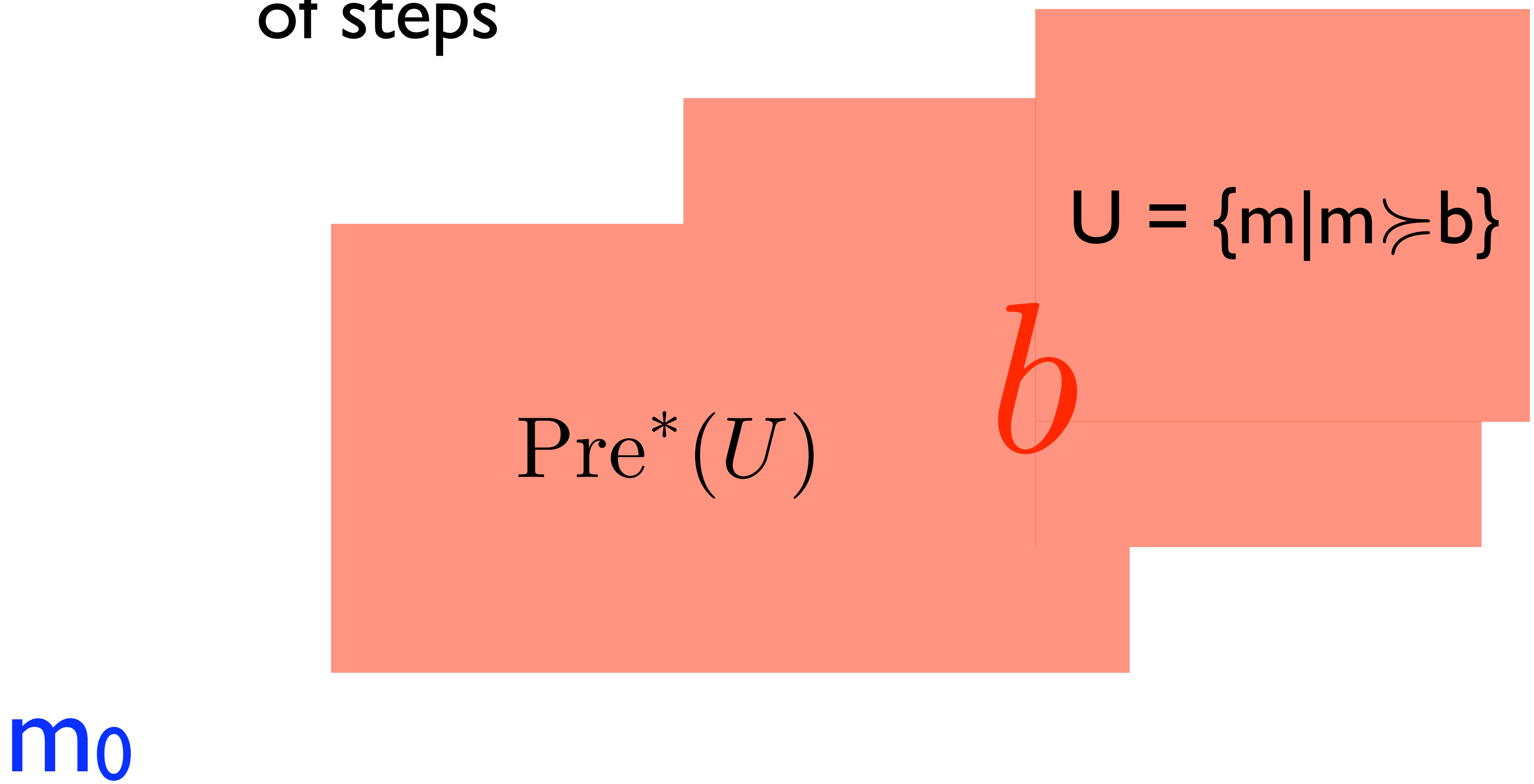


# Backward Approach

- Clearly:

  $m_0$ is in $Pre^*(U)$ iff $Reach(N) \cap U \neq \phi$

- Question: can we compute $Pre^*(U)$ ?
  - Yes !

# Predecessor operator

- Symmetrically to the Post, we define the predecessor operator:

$$\mathrm{Pre}(m) = \{m' \mid m \text{ is in } \mathrm{Post}(m')\}$$

- Let us consider the sequence

  U, Pre(U), Pre(Pre(U)), Pre(Pre(Pre(U))),...

- Theorem: After a finite amount of steps, the sequence stabilises, and we obtain $\mathrm{Pre}^*(U)$


# Advertisement

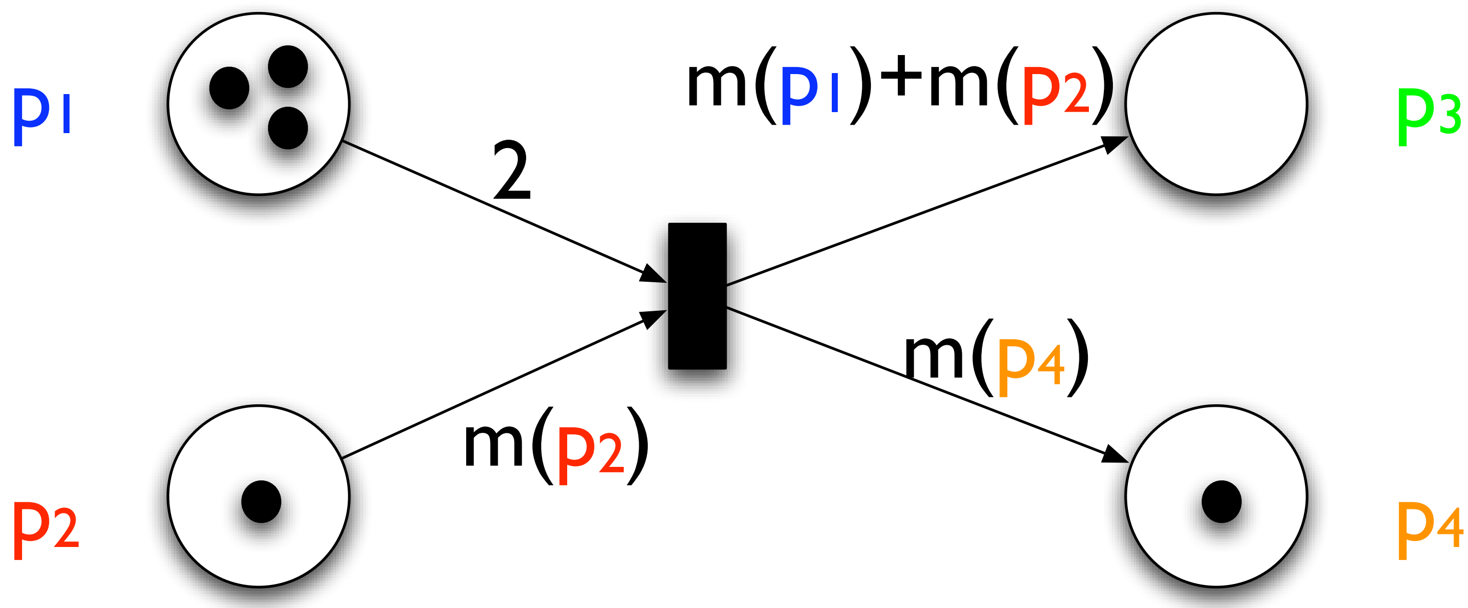

- Efficient datastuctures to implement this algorithm have been defined by researchers of the verification group at ULB.

# More on Petri nets

# Marking dependent effects

# Marking-dependent effect

- The effect of a transition is not constant anymore, but depends on the current marking.



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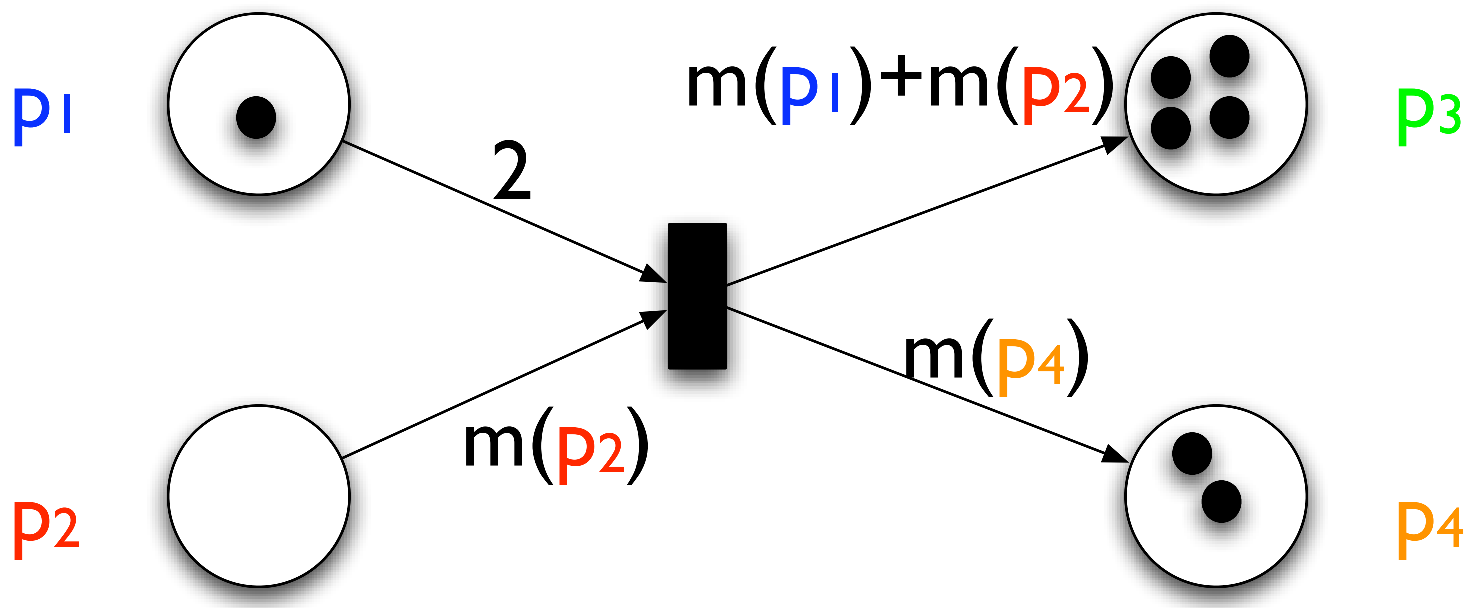
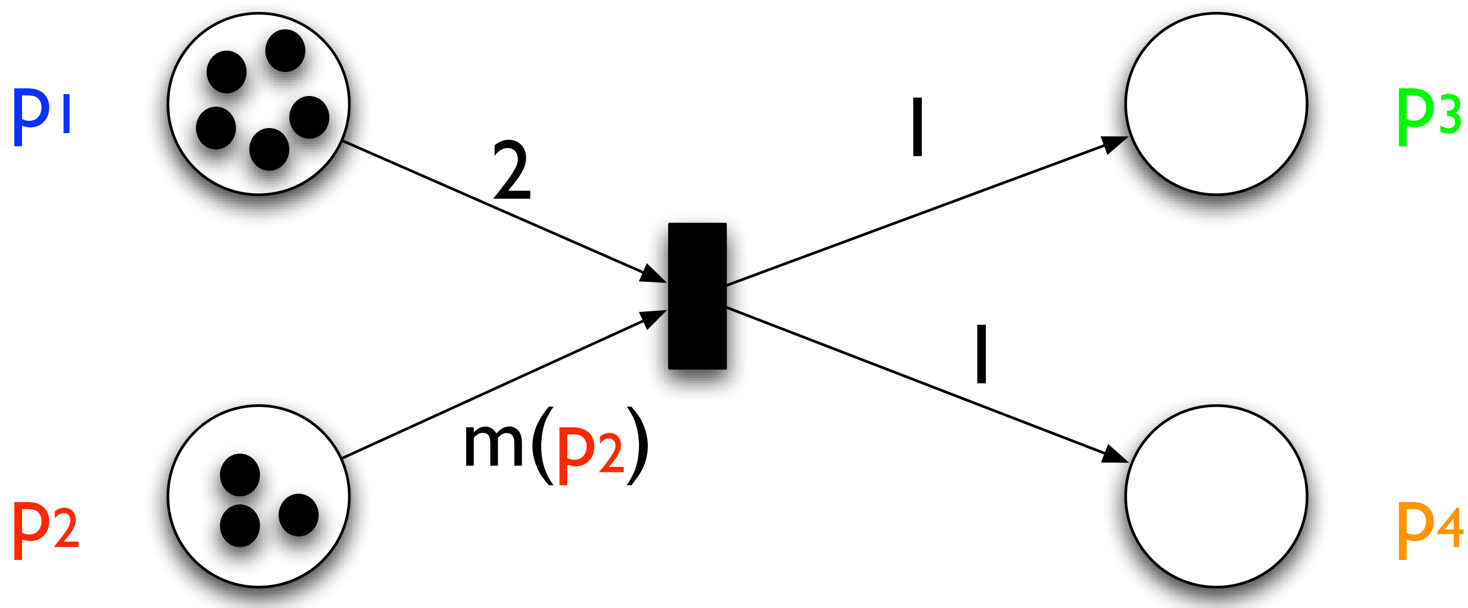


# Marking-dependent effect - resets

- In particular, we can define resets.




reset of p2

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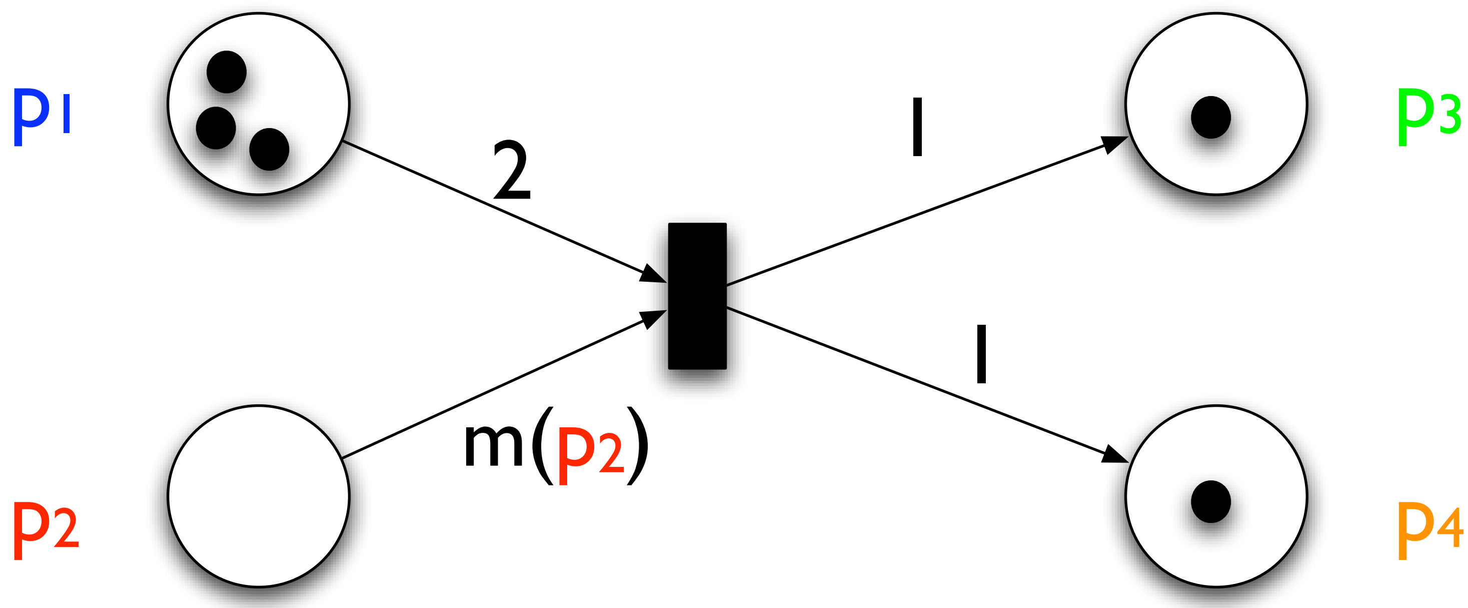


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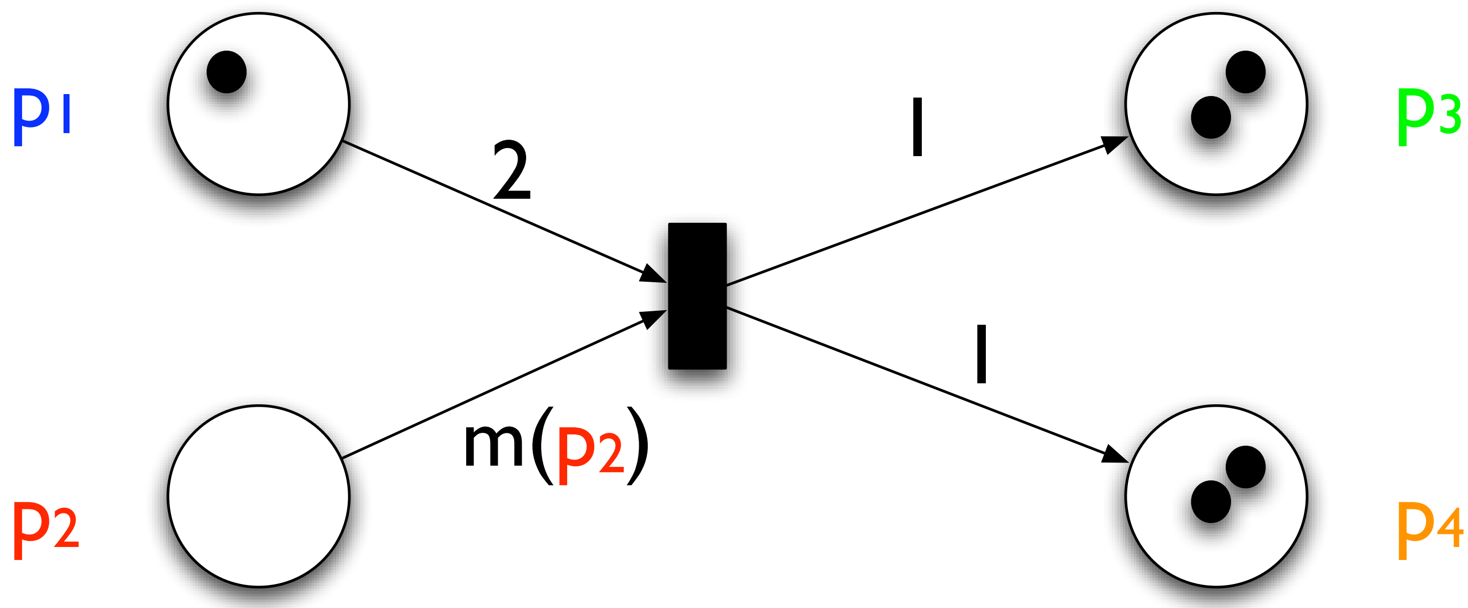


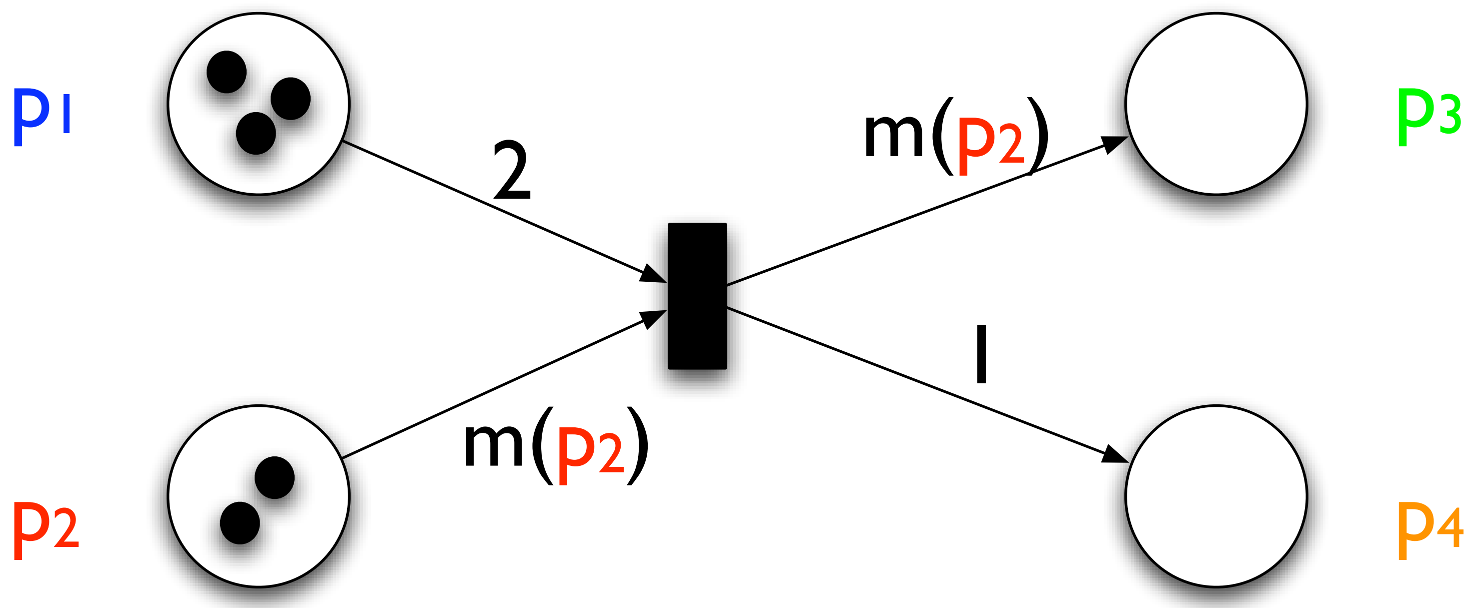

reset of p2

# Reset nets

- When we have only classical PN transitions + resets:
  - Coverability is decidable
  - Boundedness is decidable
  - Place boundedness is undecidable
  - The coverability set is not computable

# Marking-dependent effect - transfers

- In particular, we can define transfers.




transfer from $p_2$ to $p_3$

# Marking-dependent effect - transfers

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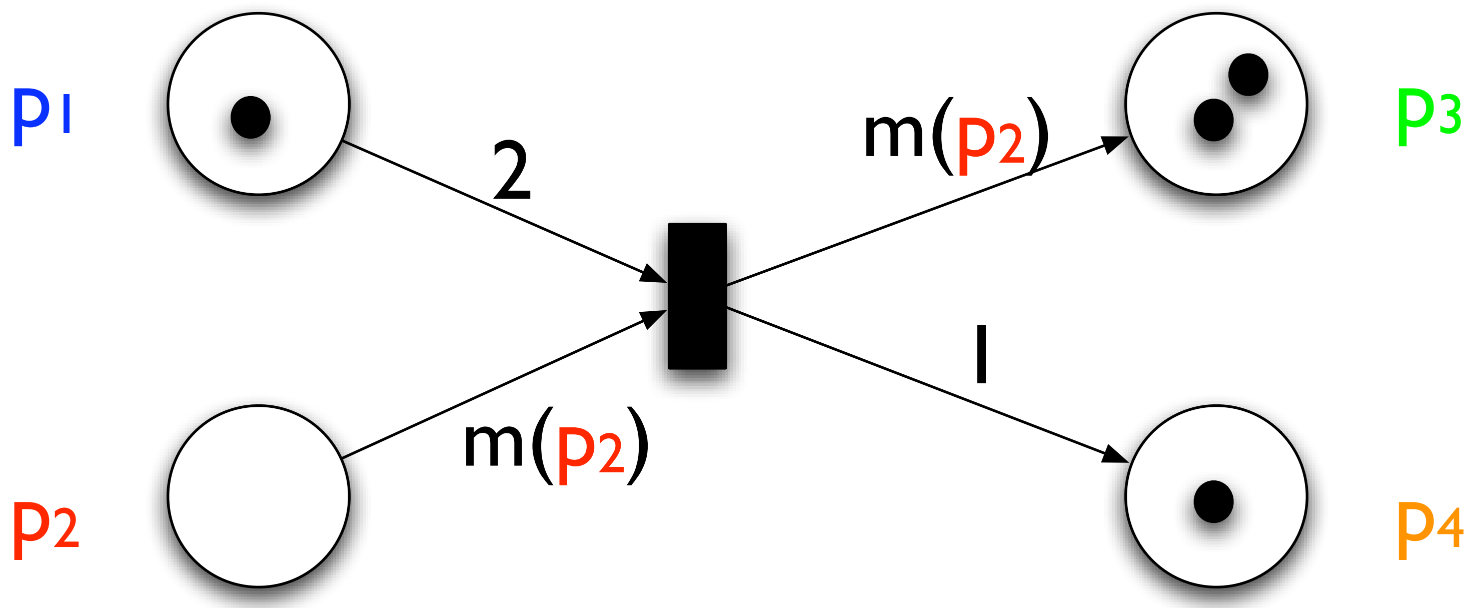
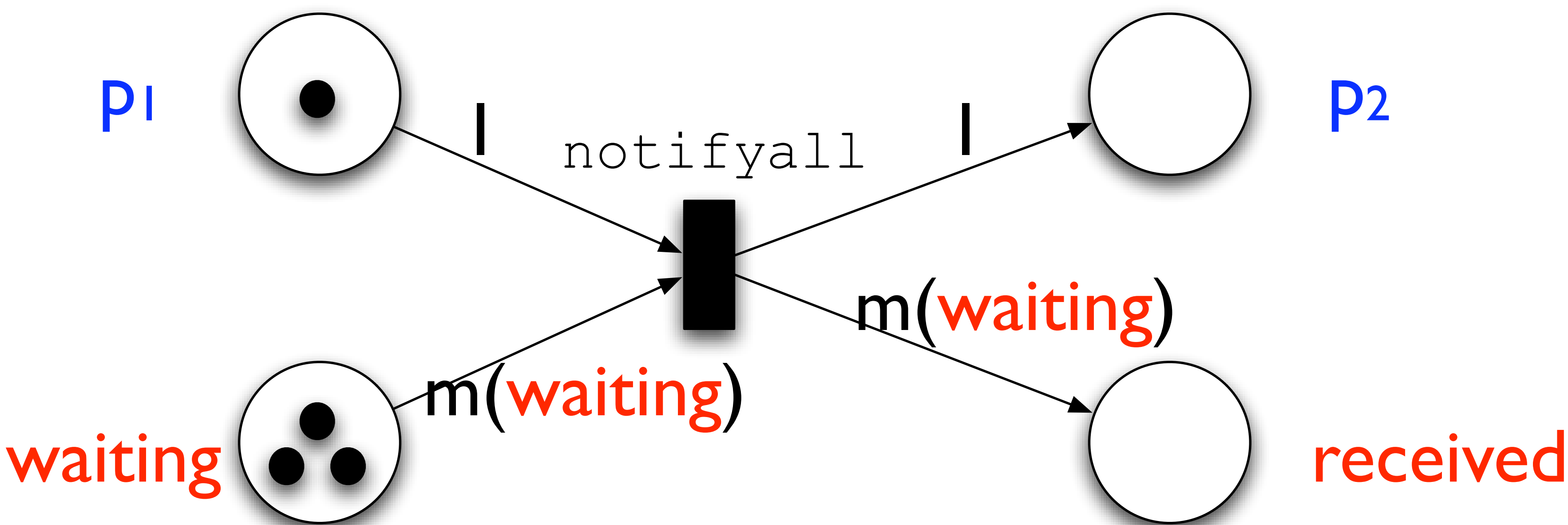



transfer from p2 to p3

# Usefulness of transfers

- Modelisation of broadcasts :
  - A single message is sent to every process
  - Each process that receives the message moves to another state.



# Transfer nets

- When we have only classical PN transitions + transfers:
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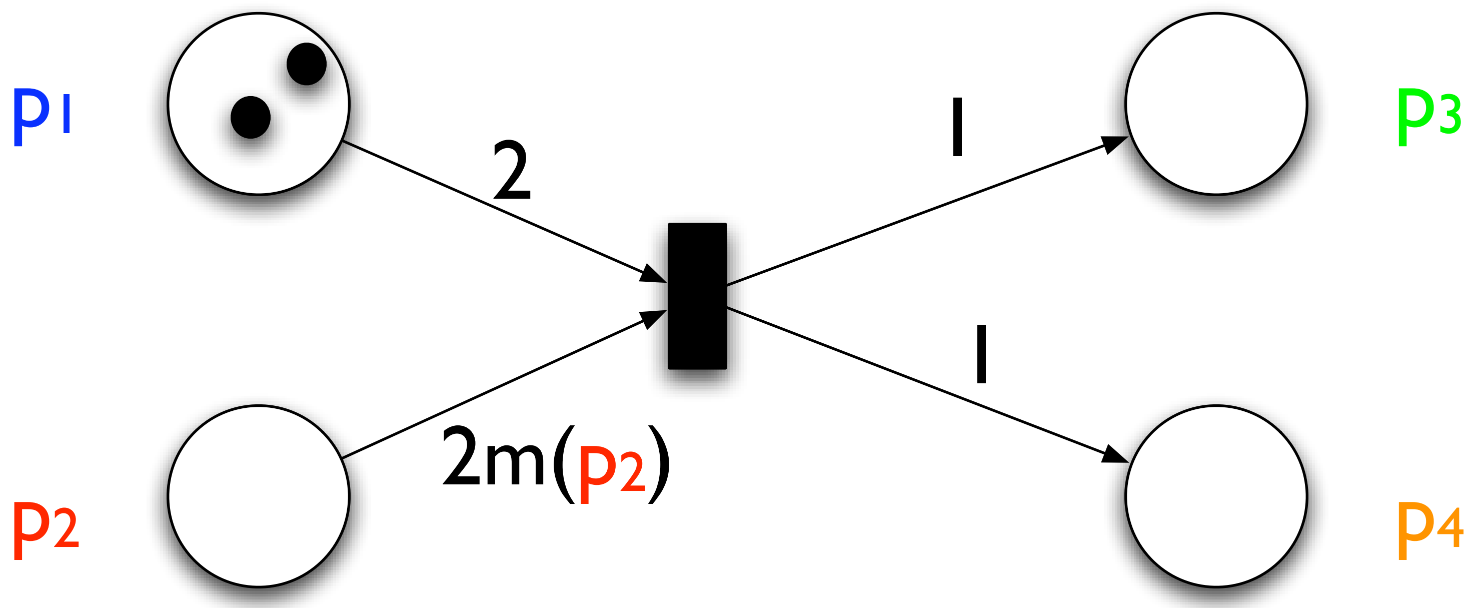
# Marking-dependent effect - zero-test

- In particular, we can define test for zero.




enabled only if p2 is empty

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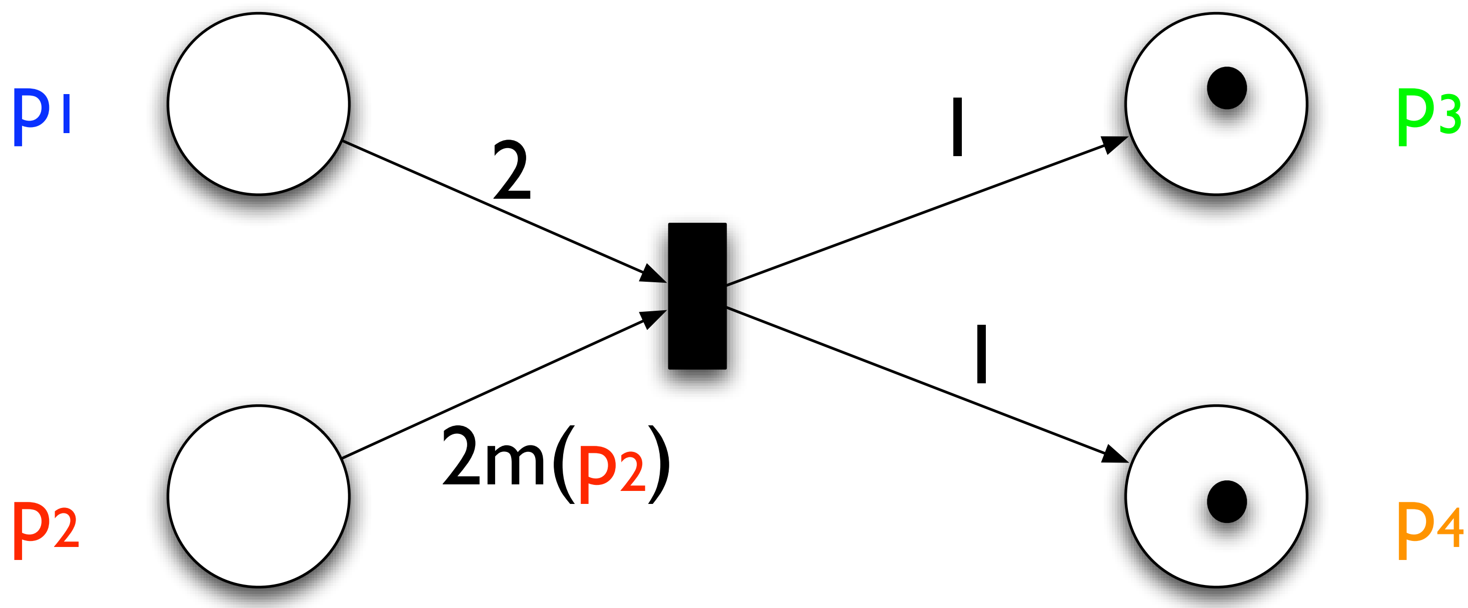


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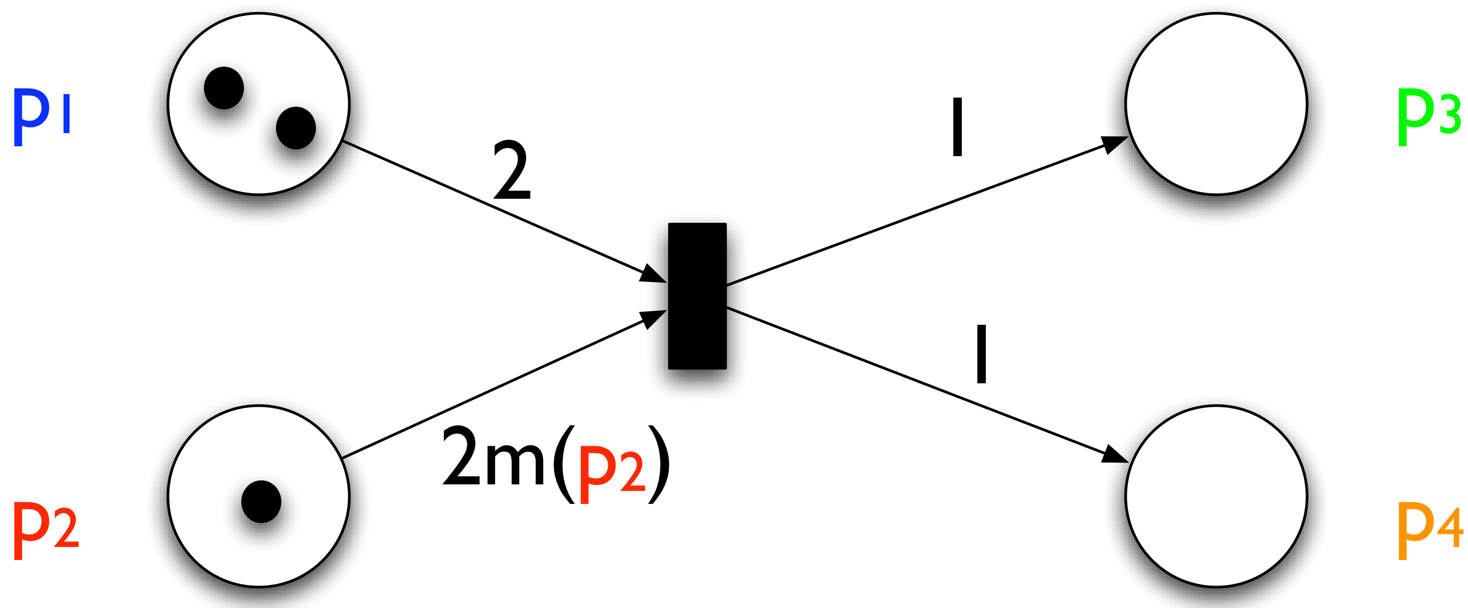



enabled only if p2 is empty

# Test for zero

- Once we have test-for-zero everything becomes undecidable.

# Coloured Petri nets

# Coloured Petri nets

- Popular extension of the basic model.
  - Introduced by the team of Kurt Jensen, in the ‘80s
  - used in many applications

# Coloured Petri nets

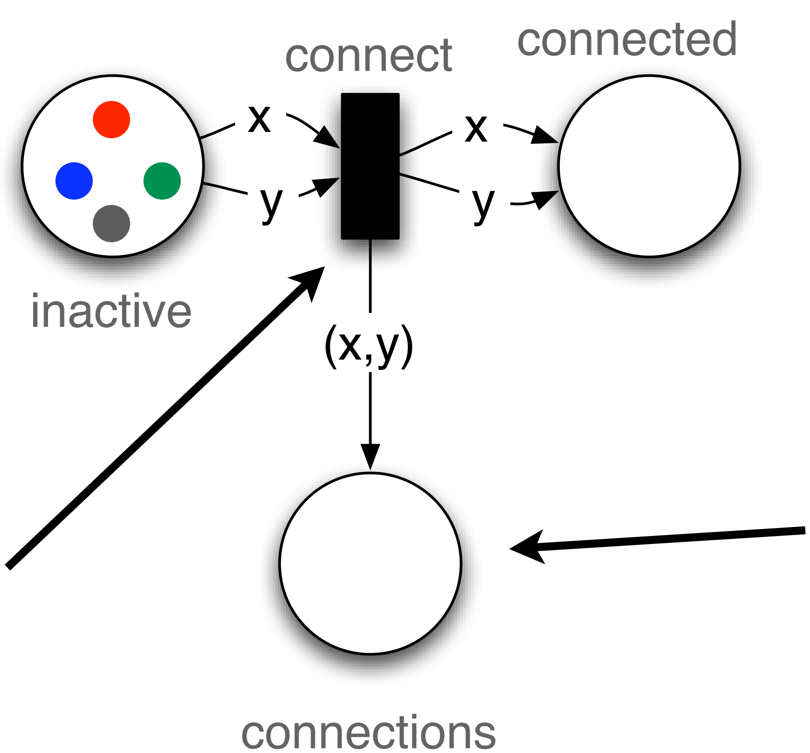
- Idea: add colours to the tokens
  - Allow to distinguish between different types of tokens
  - The colours can model data carried by the processes
  - Transitions are aware of the colours

# Phone example

- We have a set of customers:
  - Each customer is represented by a token.
  - Color of the token = Phone number.
  - A customer is either inactive or connected.

# Phone example

- A pair of inactive customers can establish a connection.
  - We want to distinguish between sender and receiver.




The transition consumes a sender x and a receiver y

Connections are recorded here as tokens whose color is a pair (snd, rcv)

# Phone example

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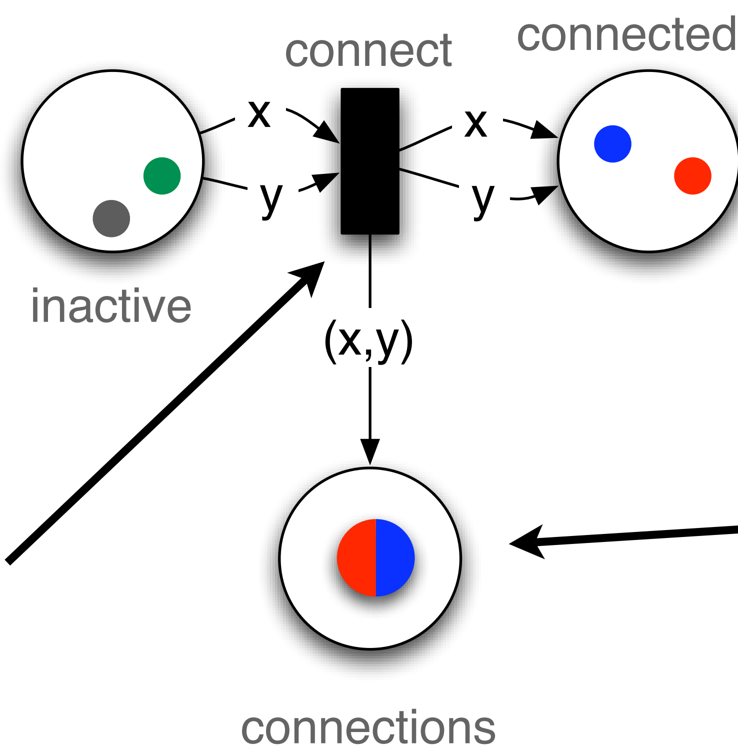


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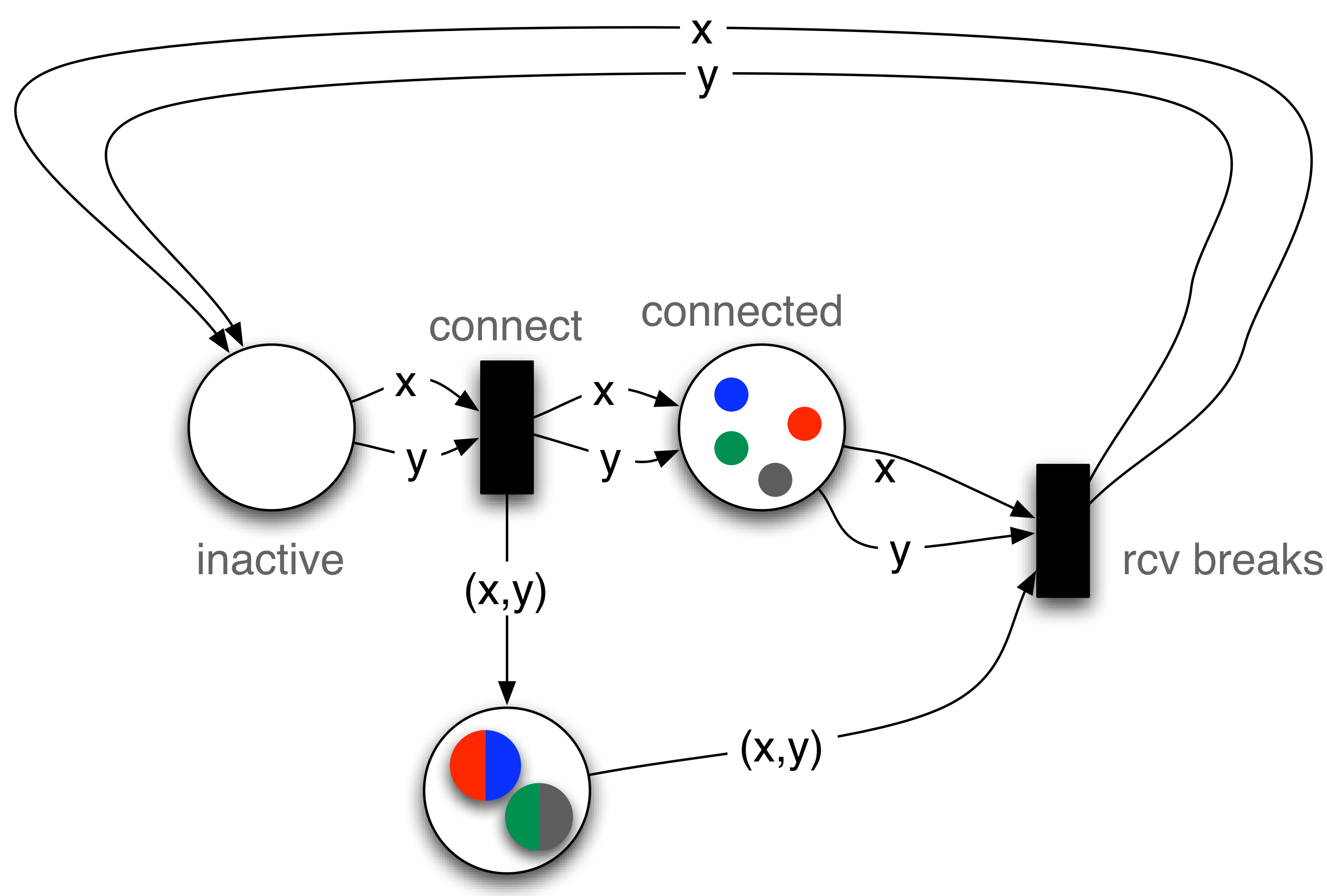
# Phone example

- The connection can be closed either by the sender or by the receiver.



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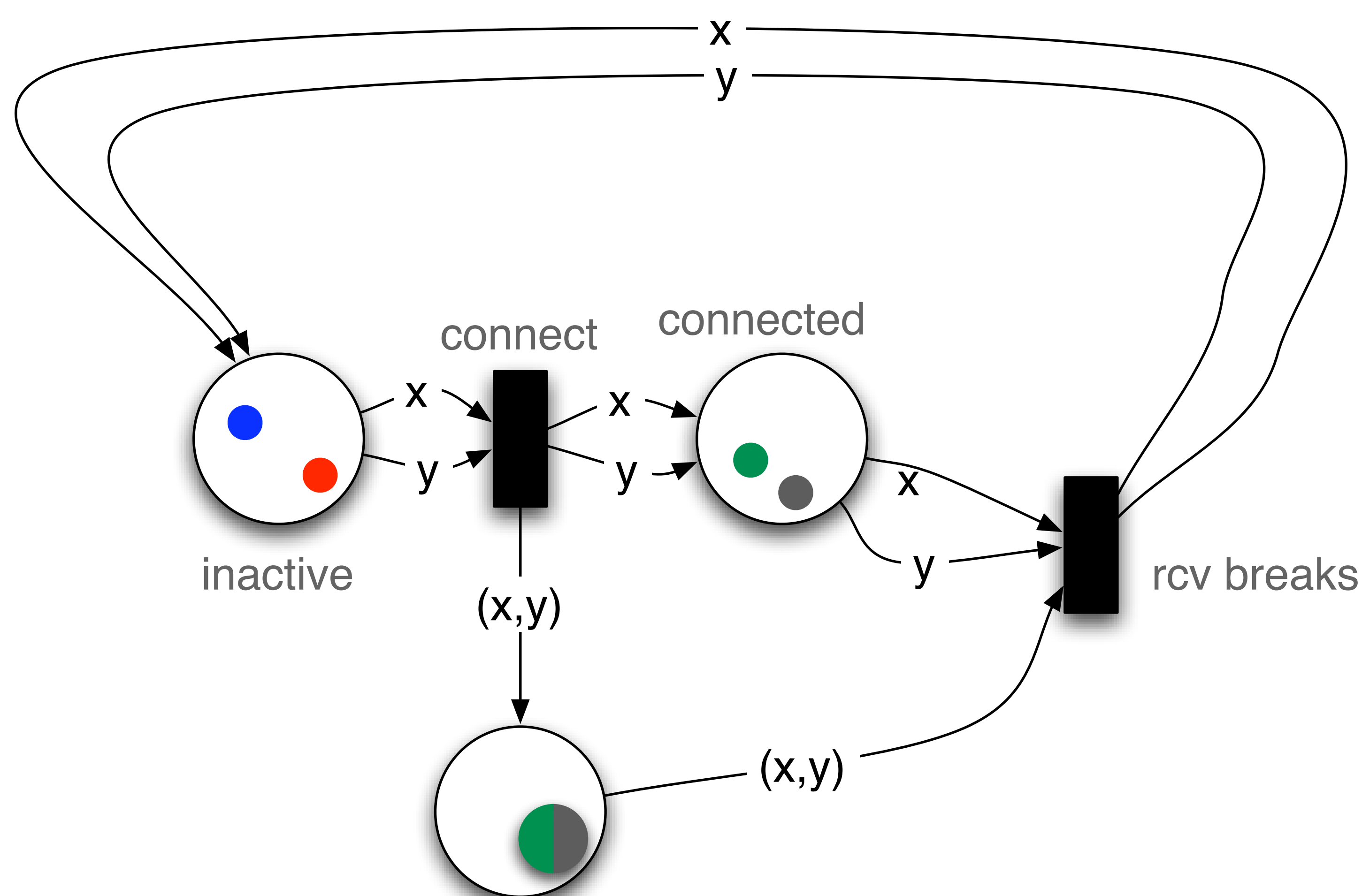

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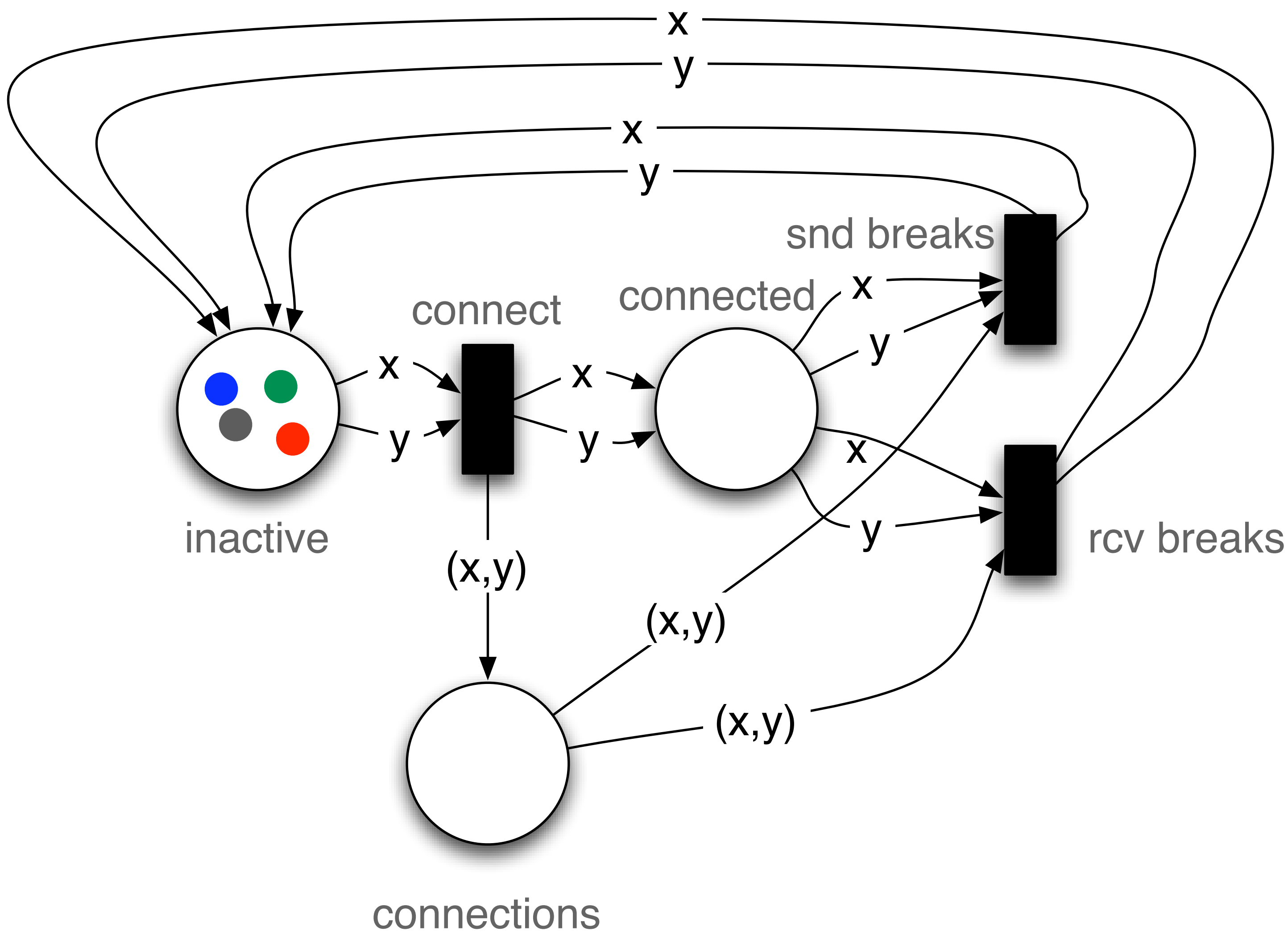
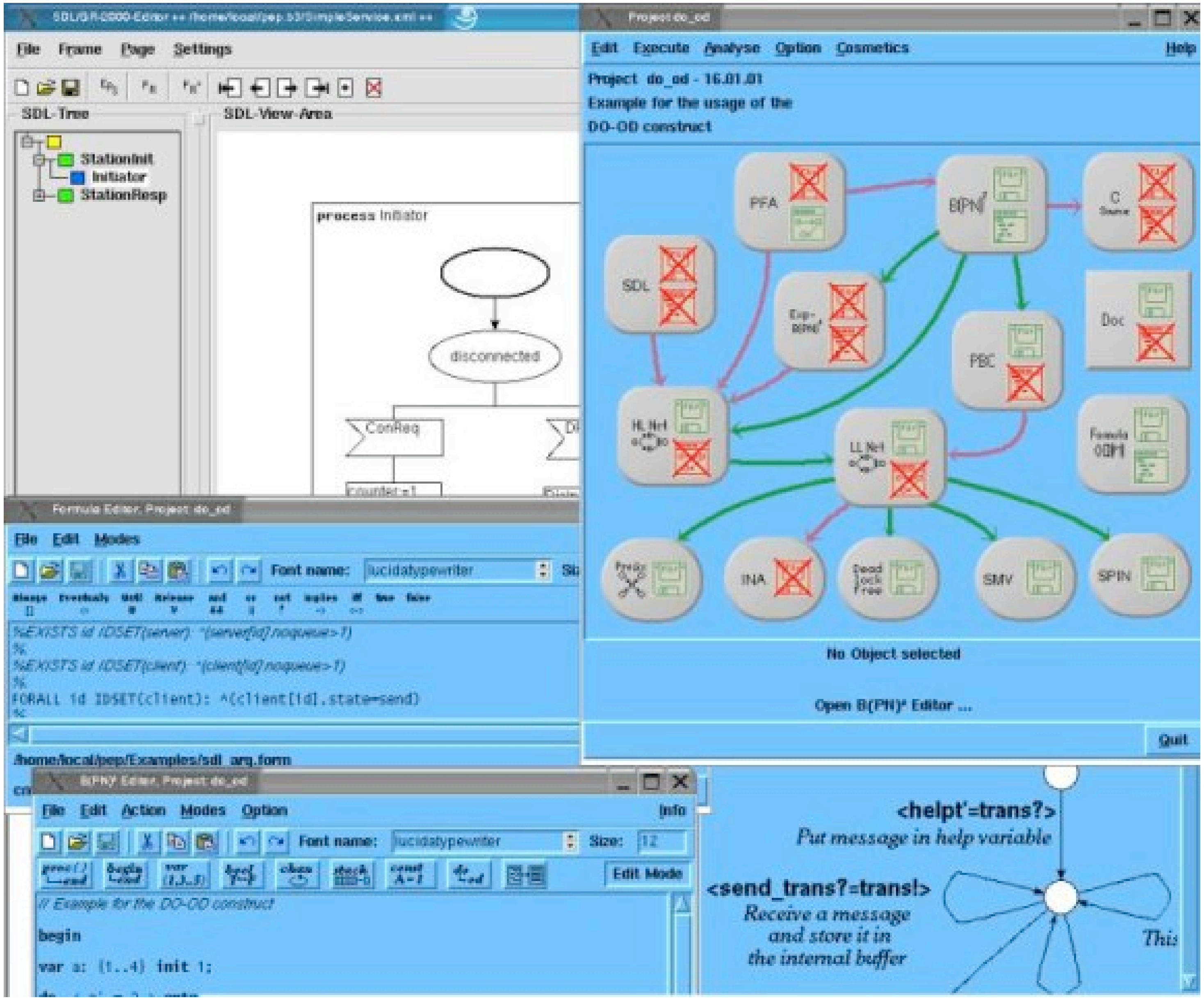


# Coloured Petri nets

- Several analysis methods have been developped for this model (finite number of colours)
    - e.g.: invariants
- Some results can be achieved when the colors have good properties

# Tools

# Practical Tools: Pep

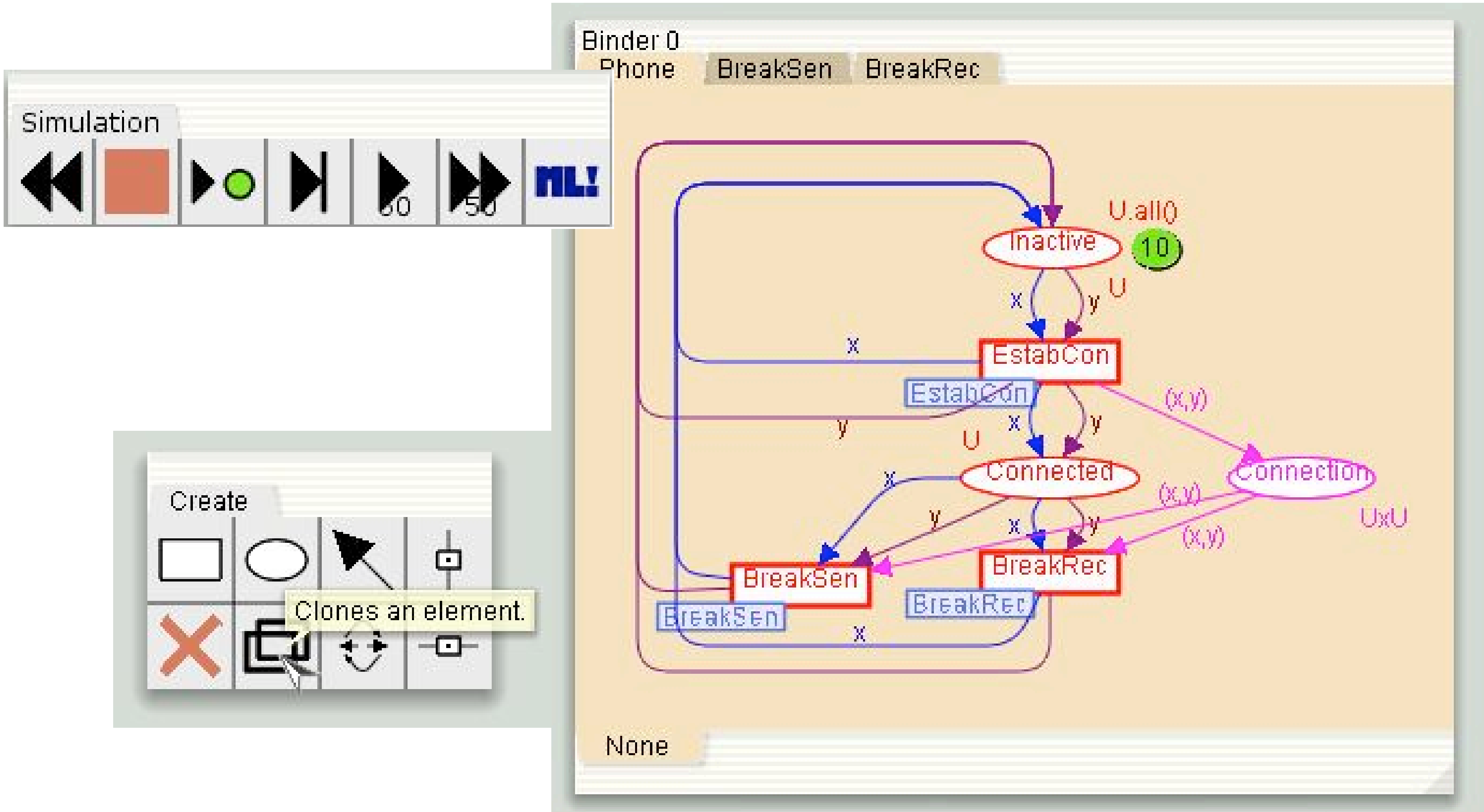


# Practical Tools: Pep

- = language to describe PN + a suite of tools to analyse them:
  - simulation
  - verification (SPIN, SMV)
  - translation from/to different formalisms
  - ...
- Everything can be accessed through a single graphical interface (Tcl/Tk)

http://theoretica.informatik.uni-oldenburg.de/~pep/

# Practical Tools: CPNTools



# Practical Tools: CPNTools

- Specialised in Coloured Petri nets
- Features similar to Pep:
  - modelisation
  - simulation
  - state space analysis
  - ...

http://wiki.daimi.au.dk/cpntools/cpntools.wiki

# Conclusion

# To conclude

- Petri nets (and their extensions) are a nice tool to reason about concurrent systems:
  - very popular
  - non-trivial decision problems are decidable
  - appealing graphical representation
  - tool supported

# To conclude

- There is still a lot to explore:
  - other extensions:
    - Time Petri nets
    - Timed Petri nets
    - Stochastic Petri nets,...

# To conclude

- There is still a lot to explore:
  - Subclasses of Petri nets:
    - 1-safe
    - marked graphs
    - free-choice
    - conflict free
    - ...
  - Some problems are easier to decide on these subclasses.

# To conclude

- There is still a lot to explore:
  - other problems:
    - liveness
    - deadlock freedom
    - semi-linearity
    - non-termination
    - ...

# To conclude

- Very active field of research !
  - Several conference and journals entirely dedicated to Petri nets
  - ... just hop in and join us !

http://www.informatik.uni-hamburg.de/TGI/PetriNets/

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- More at:
  - http://www.informatik.uni-hamburg.de/TGI/PetriNets/introductions/

# Questions ?