Specification and Computation of Word Transductions

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Université libre de Bruxelles & FNRS

Workshop 'Trends in Transformations', 2018

Specification

Describe properties

Computation

Decide those properties

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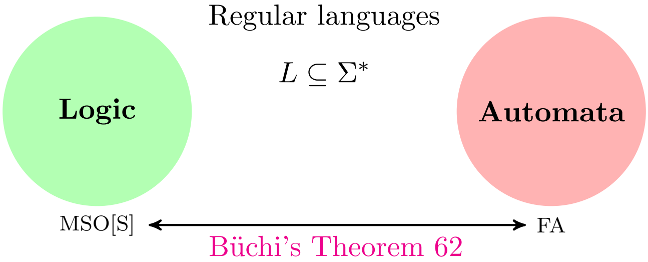
Decide those properties



Regular languages

$L \subseteq \Sigma^*$

Logic

Automata

MSO[S]

FA

Büchi's Theorem 62

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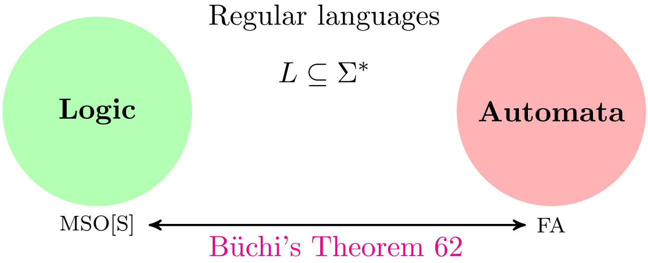

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Many extensions: infinite words, finite and infinite trees, graphs, other logics ...

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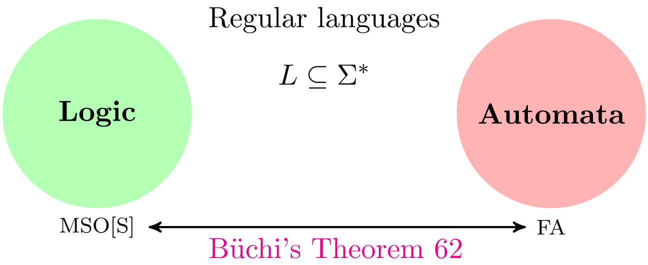

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Famous application: Model-checking $A \models \phi$?

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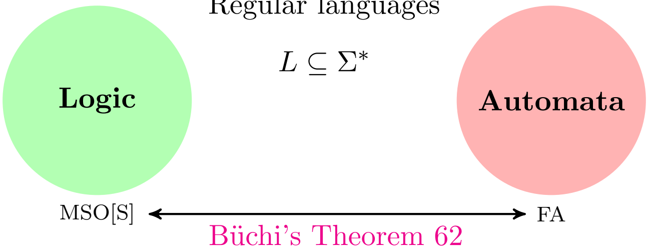

Many extensions: infinite words, finite and infinite trees, graphs, other logics ...

Famous application: Model-checking $A \models \phi$?

$L(A) \cap L(A_{\neg\phi}) = \varnothing$

Objective of the talk

What about transductions ?

$$f : \Sigma^* \hookrightarrow \Sigma^*$$

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Mirror the input word	$trends18 \mapsto 81sdnert$

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Mirror the input word	$trends18 \mapsto 81sdnert$
Copy the input word	$krishna \mapsto krishnakrishna$

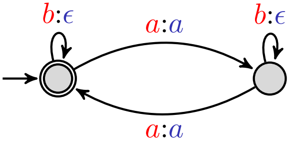
Outline

1. automata for transductions
2. logics for transductions
3. recent results

Automata models for transductions

Automata for transductions: transducers

f_{del} :



Automata for transductions: transducers

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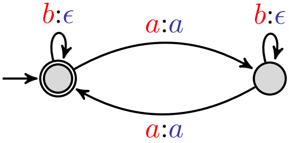

$aabaa \mapsto aaaa$

Automata for transductions: transducers

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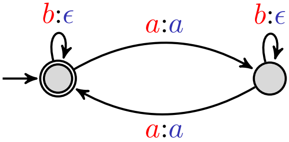

$$aabaa \mapsto aaaa$$

$$aaba \mapsto \text{undefined}$$

Automata for transductions: transducers

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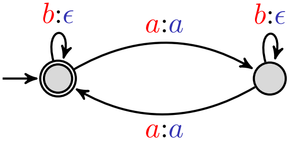
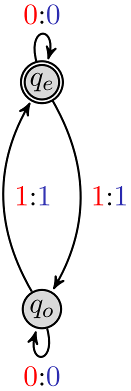


$$\begin{array}{rcl} aabaa & \mapsto & aaaa \\ aaba & \mapsto & \text{undefined} \\ dom(f_{del}) & = & \text{'even number of } a\text{'} \end{array}$$

Parity bit

$01101 \mapsto \mathbf{1}01101, 01111 \mapsto \mathbf{0}01111$



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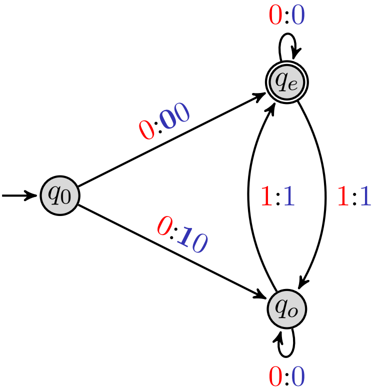

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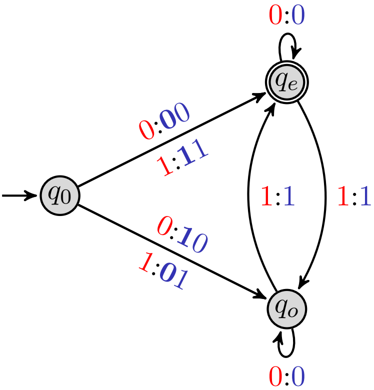

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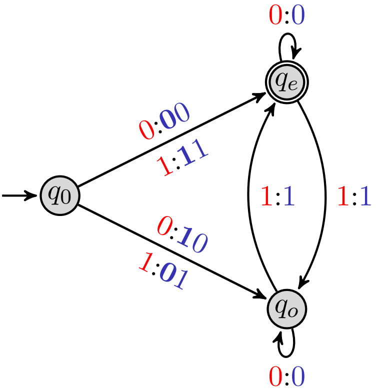

- input non-determinism needed here (aka non-sequential)
- PTIME decidable whether non-determinism is necessary

Choffrut, Sakarovitch, Carton,Beal,Prieur

Equivalence problem

Def Let $f, g : \Sigma^* \hookrightarrow \Sigma^*$ given by transducers T_f, T_g such that $dom(f) = dom(g)$. Decide whether $f = g$.

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Lem (Schützenberger) Inequivalence is witnessed by runs r_1, r_2 such that

(1) r_1, r_2 are over the same input
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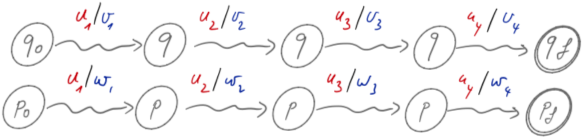
Coro Equivalence is decidable in PSpace.

Equivalence problem

Lem Proof Assume $v = f(u) \neq g(u) = w$. If u is long enough, there exists a decomposition:



with $u = u_1u_2u_3u_4$ ($|u_2| > 0$,$|u_3| > 0$), $v = v_1v_2v_3v_4$, $w = w_1w_2w_3w_4$.

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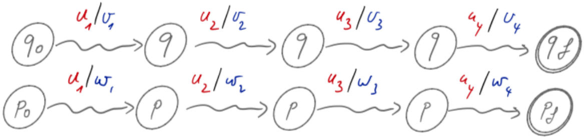

with $u = u_1 u_2 u_3 u_4$ ($|u_2| > 0$,$|u_3| > 0$), $v = v_1 v_2 v_3 v_4$, $w = w_1 w_2 w_3 w_4$. Show one of the following cases hold:

1. $f(u_1 u_4) = v_1 v_4 \neq g(u_1 u_4) = w_1 w_4$
2. $f(u_1 u_2 u_4) = v_1 v_2 v_4 \neq g(u_1 u_2 u_4) = w_1 w_2 w_4$
3. $f(u_1 u_3 u_4) = v_1 v_3 v_4 \neq g(u_1 u_3 u_4) = w_1 w_3 w_4$

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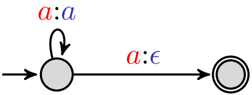
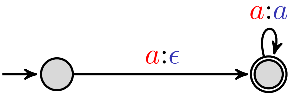
Thm (Gurari,Ibarra,83). Equivalence is decidable in PTime.

Transducer equivalence vs Automata equivalence

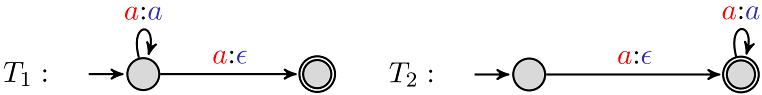
T_1 :



T_2 :



Transducer equivalence vs Automata equivalence



- Same transduction but different languages:

$$(a, \epsilon)(a, a)(a, a) \neq (a, a)(a, a)(a, \epsilon)$$

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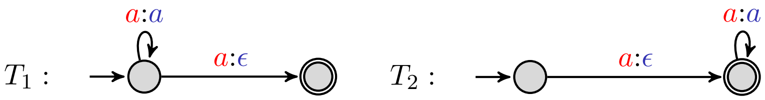


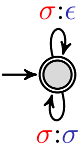
- Same transduction but different languages:

$$(a,\epsilon)(a,a)(a,a) \neq (a,a)(a,a)(a,\epsilon)$$

- Transducers are **asynchronous**
- Make most transducer problems conceptually difficult (and even computationally).

Non-determinism and relations

In general, transducers define binary *relations* in $\Sigma^* \times \Sigma^*$



realizes $\{(u, v) \mid v \text{ is a subword of } u\}$

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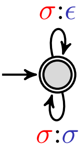

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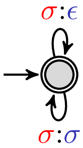

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Let $R_1, R_2 \subseteq \Sigma^* \times \Gamma^*$ given by transducers. Decide if $R_1 = R_2$.

- Undecidable (Griffith 68), even if one alphabet is unary (Ibarra 78)

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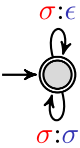

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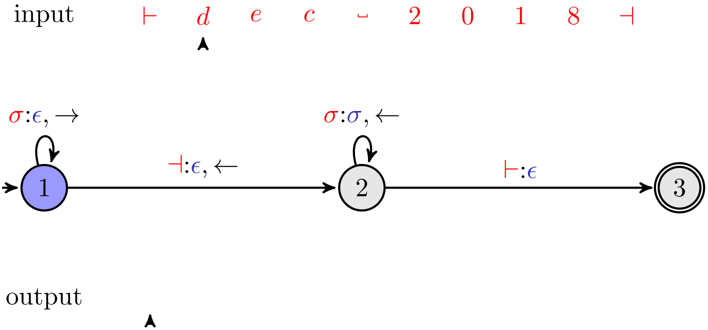
- Undecidable (Griffith 68), even if one alphabet is unary (Ibarra 78)
- Decidable for bounded-valued transducers (Culik Karhumäki 86) [1].

[1] $\exists K \forall u \; |T(u)| \leq K$

Two-way finite transducers (2FT)



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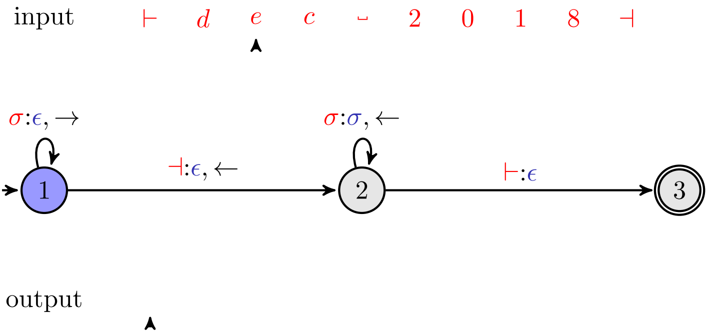

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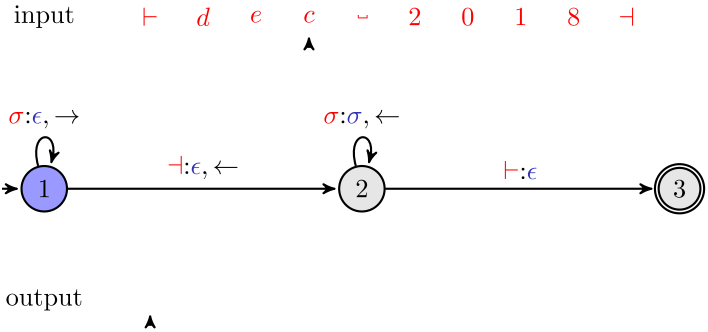
input
$\vdash$ d e c ␣ 2 0 1 8 $\dashv$
σ:ϵ, $\rightarrow$
σ:σ, $\leftarrow$
$\dashv$:ϵ, $\leftarrow$
$\vdash$:ϵ
1
2
3
output

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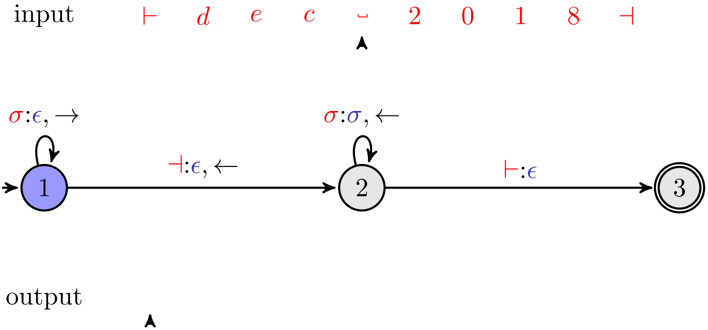
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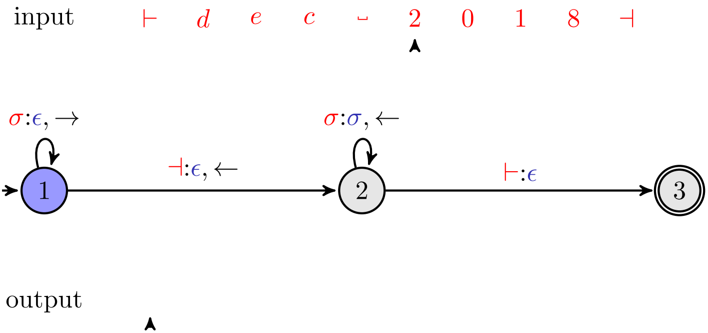

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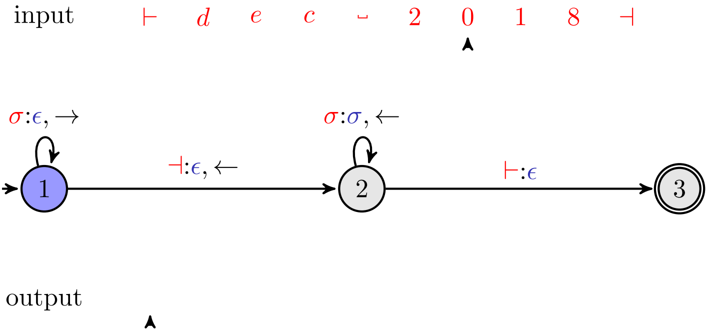

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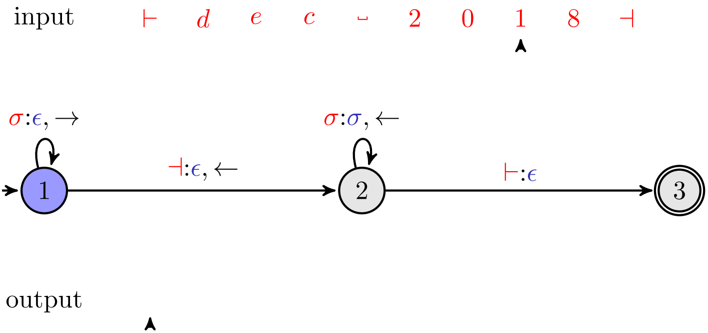
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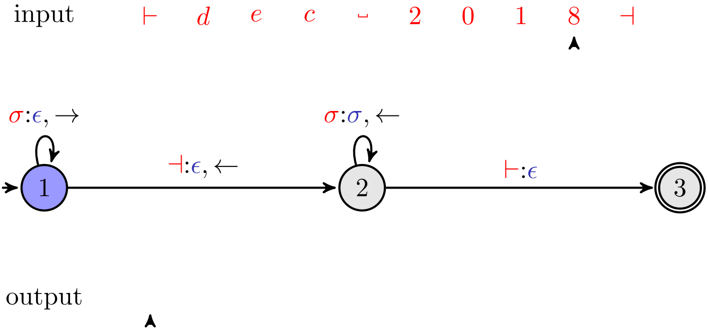

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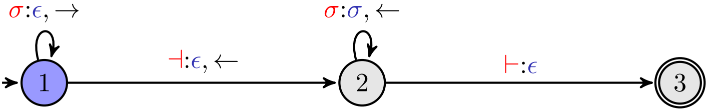
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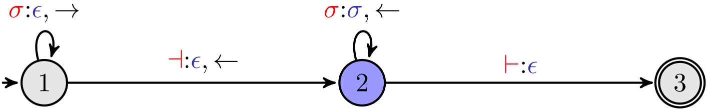

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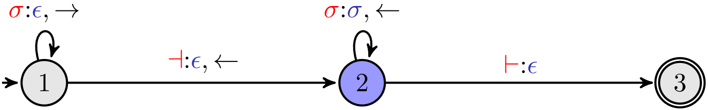
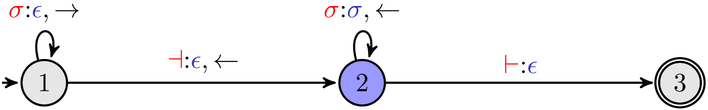


output 8

Two-way finite transducers (2FT)

input $\vdash$ d e c ␣ 2 0 1 8 $\dashv$



output 8 1

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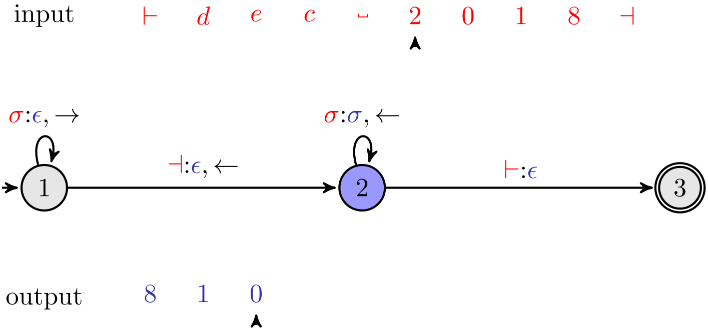

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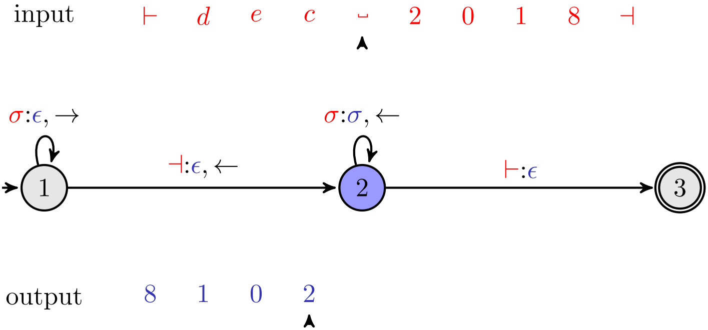

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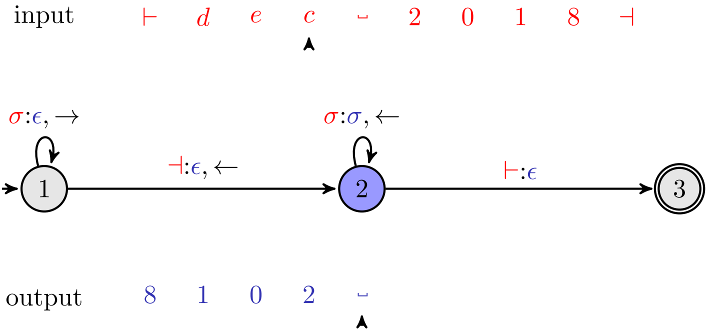

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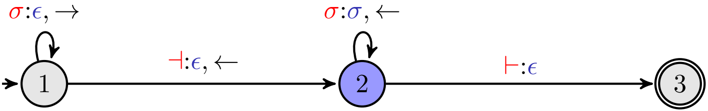
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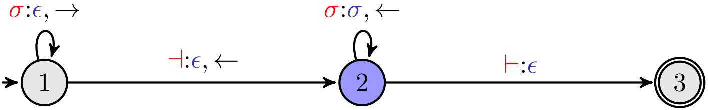

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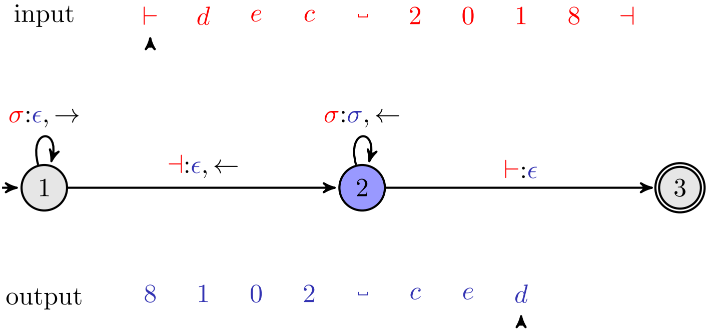

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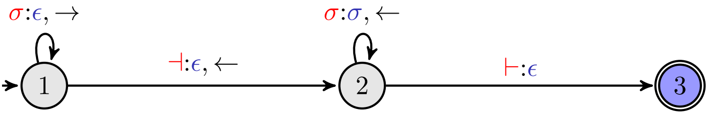

output 8 1 0 2 ␣ c e d

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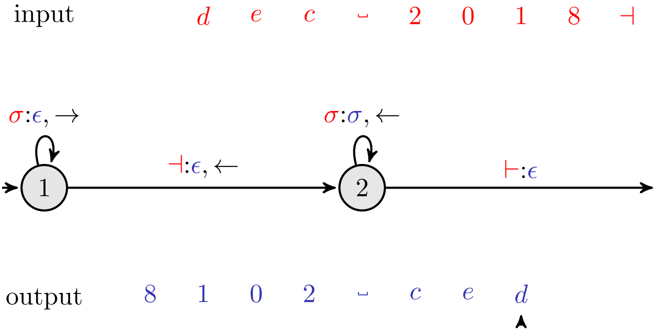
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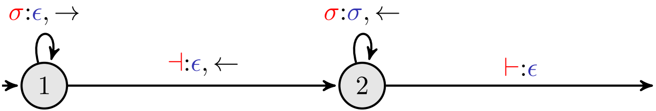
$\dashv$:ϵ, $\leftarrow$
$\vdash$:ϵ
1
2
output
8 1 0 2 ␣ c e d

Other examples

$dec, 2018 \mapsto 2018, dec$

Two-way finite transducers (2FT)

input d e c ␣ 2 0 1 8 ⊣



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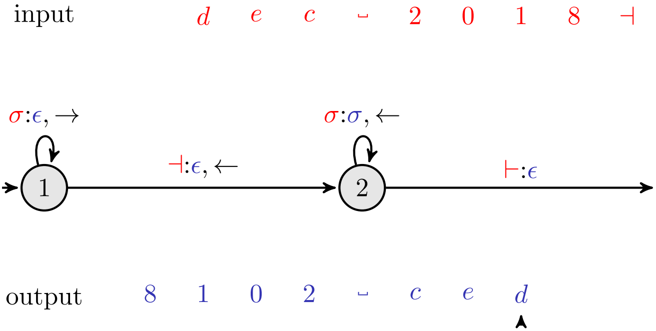
input
d e c ␣ 2 0 1 8 $\dashv$
$\sigma{:}\epsilon, \rightarrow$
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$u \mapsto f_1(u) f_2(u)$ (copy+trans)

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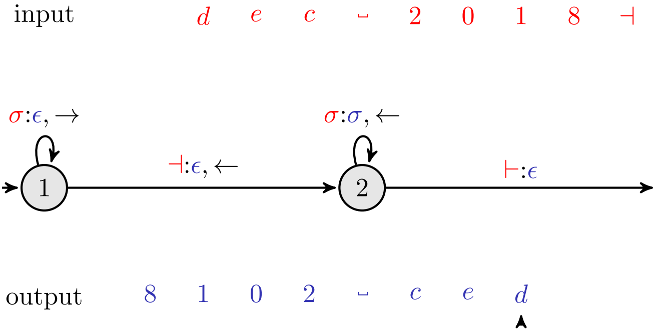

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$u_1\#u_2\#\ldots\#u_k \mapsto \overline{u_1}\#\ldots\#\overline{u_k}$ (local reverse)

Some important results on two-way transducers

Over (functional) transductions:

- equivalence is decidable in PSPACE

  (Gurari 82) (Culik, Karhumäki,87)

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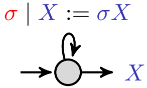
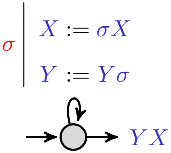
- and to many other models ...

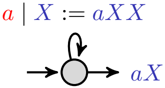
Transducers with registers



mirror



id.mirror



$exp{:}a^n \mapsto a^{2^n}$

- deterministic one-way
- equivalent to 2FT if linear updates

(Alur, Cerny, 10)

- decidable equivalence problem

(F., Reynier,17) (Benedikt et. al., 17)

Summary – Expressiveness

	input-deterministic		functional		non-functional
1-way (rational)		<		<	
	∧		∧		∧
2-way (regular)		=		<	

Summary – Expressiveness

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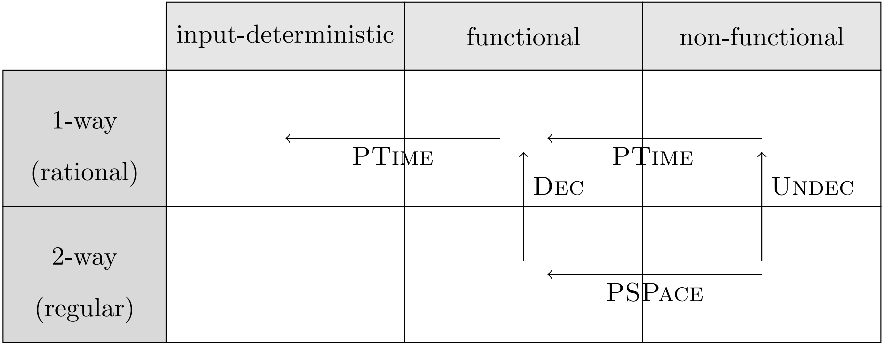

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1-way (rational)			
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PTime

PTime

Dec

Undec

PSPace

(F., Gauwin,Reynier,Servais,13)

(Baschenis, Gauwin,Muscholl,Puppis,17)

Summary – Equivalence problem

	input-deterministic	functional	non-functional
1-way (rational)	PTIME	PTIME	undec
2-way (regular)	PSPACE	PSPACE	undec

Logics for transductions

MSO on words

Over some finite alphabet Σ:

$$\varphi \ ::= \ \varphi \wedge \varphi \mid \neg\varphi \mid \exists x \varphi \mid \exists X \varphi \mid x \in X \mid \sigma(x) \mid S(x,y) \qquad \sigma \in \Sigma$$

Over finite words, (set) variables interpreted by (sets of) positions.

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Some examples

- $\leq$: transitive closure of S
- first position is an a: $a(x) \wedge \forall y(x \leq y)$
- counting modulo: odd number of a, even length, ...

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Büchi-Elgot-Trakhenbrot

A language L is MSO-definable iff it is recognisable by some finite automaton.

Extension to transductions

Example (Delete all b)

- replace input label **by** a **if it is** a
- replace input label **by** ϵ **if it is** b

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$$\phi_a(x) \equiv a(x)$$

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$$\phi_\epsilon(x) \equiv b(x)$$

Extension to transductions

Example (Append #)

- replace label σ of x by σ if x is **not** the last position

- replace label σ of x by $\sigma\#$ if x is the last position

Extension to transductions

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$$\phi_{\sigma}(x) \equiv \sigma(x) \wedge \exists y \; S(x,y)$$

- replace label σ of x by $\sigma\#$ if x is the last position

$$\phi_{\sigma\#}(x) \equiv \sigma(x) \wedge \forall y \; \neg S(x,y)$$

Extension to transductions

Example (Add a parity bit)

- replace label σ of x by 1σ if x is the first position and odd number of 1
- replace label σ of x by 0σ if x is the first position and even number of 1
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Extension to transductions

Example (Add a parity bit)

- replace label σ of x by 1σ if x is the first position and odd number of 1

$$\phi_{1\sigma}(x) \equiv \sigma(x) \wedge \forall y\ \neg S(y,x) \wedge \phi_{odd}$$

- replace label σ of x by 0σ if x is the first position and even number of 1

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Büchi Theorem for Rational Transductions

Def $f : \Sigma^* \hookrightarrow \Sigma^*$ is MSO-definable if it can be "described" by a finite set of formulas $\phi_{v_1}(x), \ldots, \phi_{v_k}(x)$ ($v_1, \ldots, v_k \subseteq \Sigma^*$).

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Proof Idea Transducers → MSO

For all transitions $t = p \xrightarrow{\sigma:v} q$ define $\phi_v^t(x)$

- which expresses the **existence of an accepting run which at position** x **triggers transition** t.
- latter property is regular, so MSO-definable (Büchi's Theorem)

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Proof Idea MSO $\rightarrow$ Transducers

- states must hold enough information to decide MSO satisfiability
- states are MSO-types (τ_1, τ_2) of (prefix,suffix)
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Büchi Theorem for Rational Transductions

What about mirror ? $csl2018 \mapsto 8102lsc$

Replace label of position x by σ if $last - x$ is labeled σ.

Not MSO-definable.

(Courcelle) MSO Transducers

“interpreting the output structure in the input structure”

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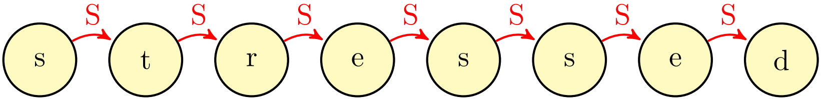

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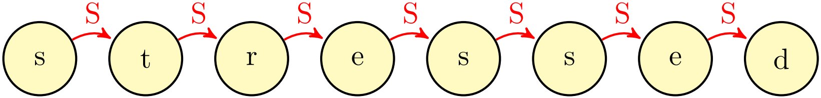

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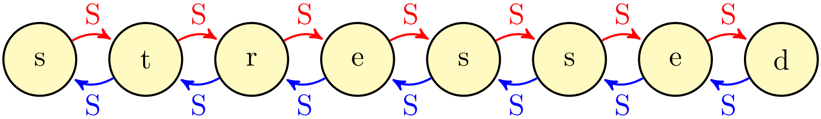

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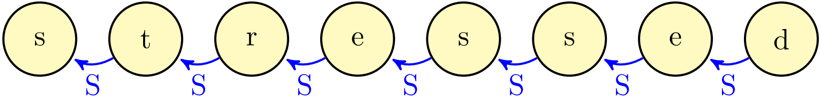

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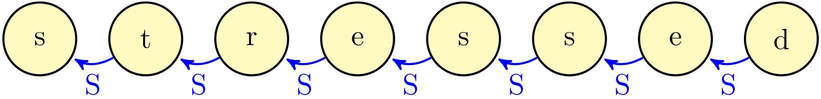

$$\phi_S(x,y) \equiv S(y,x)$$

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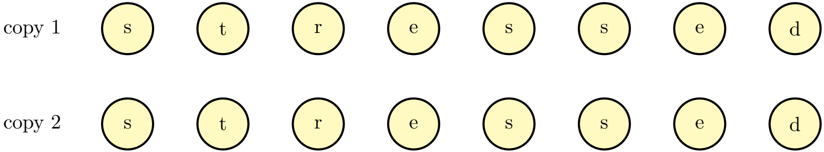
- input structure can be copied a fixed number of times: $u \mapsto uu$, or $u \mapsto u.\mathrm{mirror}(u)$.

Other example : $u \mapsto u.\mathrm{mirror}(u)$



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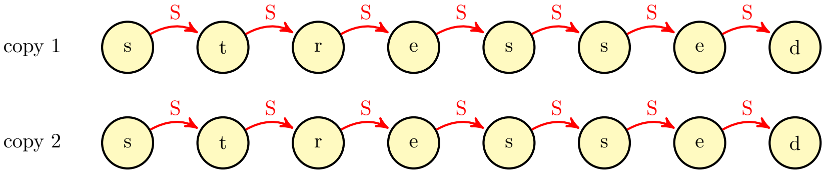

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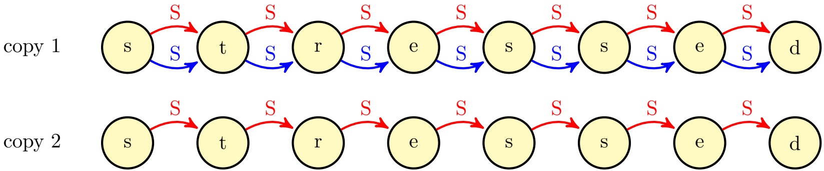

Formulas

copy 1: $\phi^{\mathbf{1}}_{S}(x,y) \quad \equiv \quad S(x,y)$

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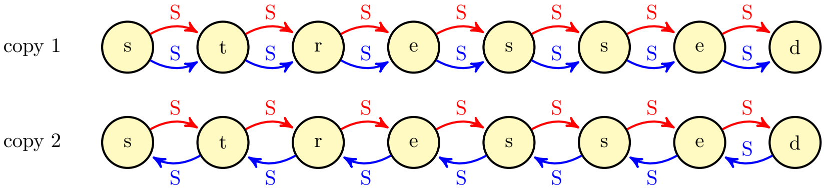

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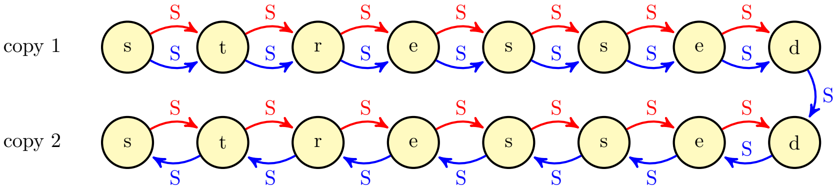

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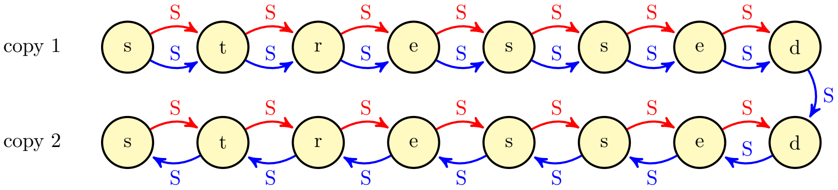

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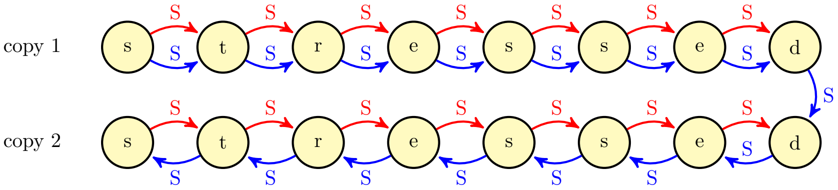

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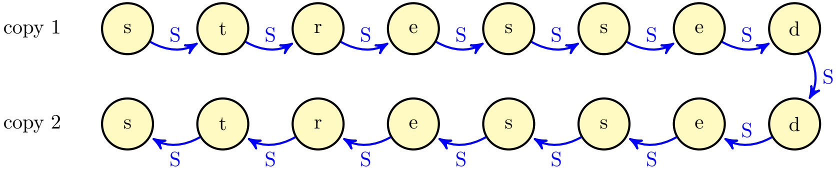

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Let $f : \Sigma^* \hookrightarrow \Sigma^*$.

Theorem (Engelfriet, Hoogeboom, 01)

The following are equivalent:

1. *f is definable by a deterministic two-way transducer*
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Proof ideas: MSO-transducers are 2-way transducers with MSO jumps $\phi_S^{c \to c'}(x, y)$

- turn jumps into walks
- hold enough information to decide MSO-formulas locally: states = MSO-types

$f = \hat{f} \circ f_{types}$ (use composition closure of 2-way trans)

Some other (recent) results

Other specification languages

- FO-transducers
  - equivalent to aperiodic transducers with registers (F., Krishna, Trivedi, 14)
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- an expressive decidable logic tailored to (non-functional) transductions Dartois, F., Lhote, 18.

Definability Problems

Definition

$\mathcal{F}$: logical fragment of MSOT (e.g. FOT)

Input: T an MSOT

Output: Is $[\![T]\!]$ FO-definable ?

Definability Problems

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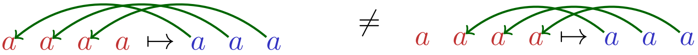
Results

- Decidable for “rational” MSO (=rational functions)
  F., Gauwin, Lhote, 16
- Open for MSOT

Origin semantics (Bojanczyk, 14)



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Origin semantics $[\![T]\!]_o$ inherent to most transducer models T !

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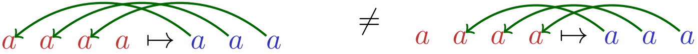

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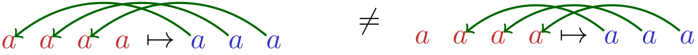

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- extended to "similar" origins through resynchronisers (F., Maneth, Reynier, Talbot, 15) (F., Jecker, Löding, Winter,16), (Bose, Muscholl, Penell, Puppis, 18)

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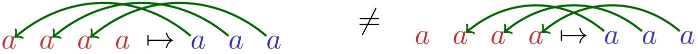

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- study of rational relation subclasses by control languages $REL(C)$, $C \subseteq \{\mathrm{in}, \mathrm{out}\}^*$ (Descotte, Figueira, Libkin, Puppis)

Some Other Results

- machine-independent characterisations

  Cadilhac,Krebs,Ludwig,Paperman, 15

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  Boiret, Lemay, Niehren 12

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- other structures: infinite strings, nested words, trees, graphs, data words ...

A Few Applications

- language and speech processing (M. Mohri)
- regular model-checking
- text analysis, document transformation
- reactive synthesis
- **Tools**: OpenFST, Vaucanson, DreX (Alur, d'Antoni, Raghothaman)
- line of works on symbolic transducers (d'Antoni, Veanes ...)

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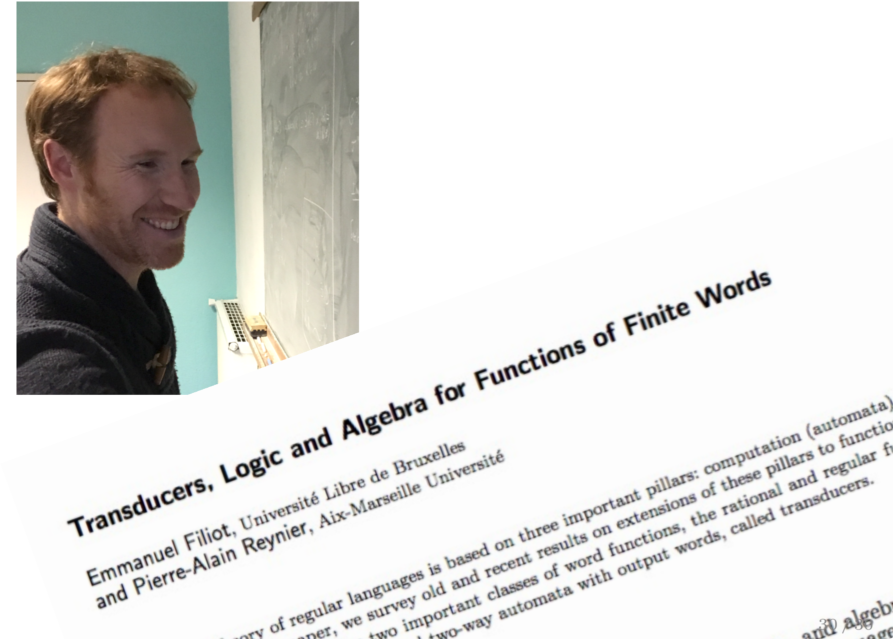
But **so far**, no strong application of the "asynchronous" setting (non letter-to-letter)

People

From left-to-right top-to-bottom: Nathan Lhote, Ismaël Jecker, Luc Dartois, Olivier Gauwin, Anca Muscholl, Frédéric Servais, Ashutosh Trivedi, Léo Exibard, Jean-Marc Talbot, Krishna S., Nicolas Mazzocchi, Christof Löding, Sarah Winter.

SIGLOG News 9th



Transducers, Logic and Algebra for Functions of Finite Words

Emmanuel Filiot, Université Libre de Bruxelles
and Pierre-Alain Reynier, Aix-Marseille Université

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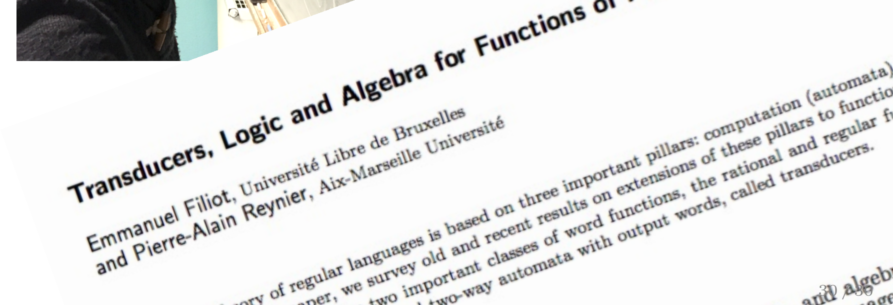

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