The Expressiveness of Metric Temporal Logic II:

This time it's irrational!

Paul Hunter

Université Libre de Bruxelles

(Joint work with Joël Ouaknine and James Worrell)

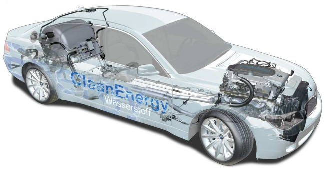
Université Libre de Bruxelles, March 2013

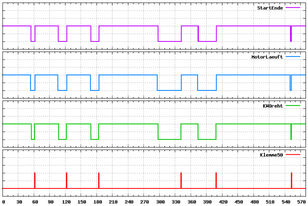
Reasoning about time

Timed systems are everywhere:

- Hardware circuits
- Communication protocols
- Cell phones
- Plant controllers
- Aircraft navigation systems
- ...

Reasoning about time





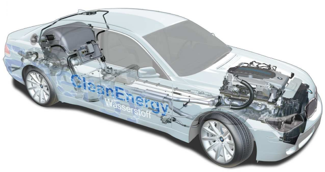
Want to specify:

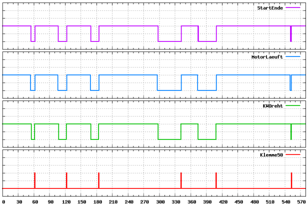
“*If I press the brake pedal then the pads will be applied*.”

“*If the brakes are applied then the pedal has been pressed*.”

Expressiveness *vs* **Computability**

Reasoning about time





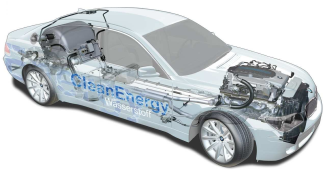
Want to specify:

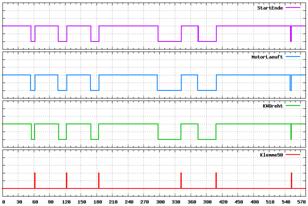
"If I press the brake pedal then the pads will be applied."

"If the brakes are applied then the pedal has been pressed."

Expressiveness *vs* Computability

Reasoning about time





Want to specify:

"If I press the brake pedal then the pads will be applied."

"If the brakes are applied then the pedal has been pressed."

Expressiveness *vs* **Computability**

Classic temporal logic

Metric temporal logic

Extending Kamp’s Theorem (again)

Temporal models

- A set **MP** of propositions: $P, Q, R, \ldots$
- Continuous time model: $\mathbb{R}$

Temporal models

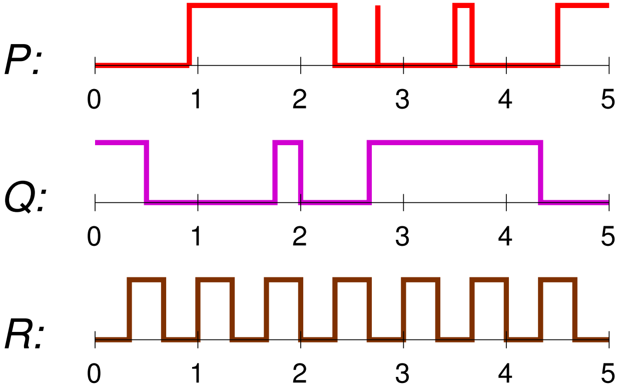
- A set **MP** of propositions: $P, Q, R, \ldots$
- Continuous time model: $\mathbb{R}$

$$f : \mathbb{R} \to 2^{\mathbf{MP}} \quad \text{(flow or signal)}$$

Temporal models

- A set **MP** of propositions: P, Q, R, ...
- Continuous time model: $\mathbb{R}$

$$f : \mathbb{R} \rightarrow 2^{\mathbf{MP}} \quad \text{(flow or signal)}$$



Classic temporal predicate logic

FO(<): First-order logic with $<$ and monadic predicates for each proposition $P \in \mathbf{MP}$:

$$\varphi ::= x < y \mid P(x) \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi \mid \forall x\, \varphi \mid \exists x \varphi$$

For example:

$$\forall x\, .\, \mathrm{PEDAL}(x) \rightarrow \exists y\, .\, ((y > x) \wedge \mathrm{BRAKE}(y)).$$

$$\forall x\, .\, \mathrm{BRAKE}(x) \rightarrow \exists y\, .\, ((y < x) \wedge \mathrm{PEDAL}(y)).$$

Classic temporal predicate logic

FO($<$): First-order logic with $<$ and monadic predicates for each proposition $P \in \mathbf{MP}$:

$$\varphi ::= x < y \mid P(x) \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi \mid \forall x\, \varphi \mid \exists\, x \varphi$$

For example:

$$\forall x \,.\, \mathsf{PEDAL}(x) \rightarrow \exists y \,.\, \big((y > x) \wedge \mathsf{BRAKE}(y)\big).$$

$$\forall x \,.\, \mathsf{BRAKE}(x) \rightarrow \exists y \,.\, \big((y < x) \wedge \mathsf{PEDAL}(y)\big).$$

Temporal logic: LTL

Linear Temporal Logic (LTL): Propositional logic with temporal modalities:

$$
\begin{array}{lll}
\theta & ::= & P \mid \theta_1 \wedge \theta_2 \mid \theta_1 \vee \theta_2 \mid \neg\theta \\
 & \mid & \mathbf{F}\,\theta \qquad (\theta \text{ occurs in the Future}) \\
 & \mid & \mathbf{G}\,\theta \qquad (\theta \text{ occurs always (Globally)}) \\
 & \mid & \theta_1 \,\mathbf{U}\, \theta_2 \qquad (\theta_1 \text{ holds Until } \theta_2) \\
 & \mid & \mathbf{P}\,\theta \qquad (\theta \text{ occurred in the Past}) \\
 & \mid & \mathbf{H}\,\theta \qquad (\theta \text{ has always occurred (Historically)}) \\
 & \mid & \theta_1 \,\mathbf{S}\, \theta_2 \qquad (\theta_1 \text{ has held Since } \theta_2)
\end{array}
$$

For example,

$$\mathbf{G}\,(\text{PEDAL} \rightarrow \mathbf{F}\,\text{BRAKE})$$

$$\mathbf{G}\,(\text{BRAKE} \rightarrow \mathbf{P}\,\text{PEDAL})$$

Temporal logic: LTL

Linear Temporal Logic (LTL): Propositional logic with temporal modalities:

$$
\begin{array}{rcll}
\theta & ::= & P \mid \theta_1 \wedge \theta_2 \mid \theta_1 \vee \theta_2 \mid \neg\theta & \\
& & \mid \ \mathbf{F}\,\theta & (\theta \text{ occurs in the Future}) \\
& & \mid \ \mathbf{G}\,\theta & (\theta \text{ occurs always (Globally)}) \\
& & \mid \ \theta_1 \,\mathbf{U}\, \theta_2 & (\theta_1 \text{ holds Until } \theta_2) \\
& & \mid \ \mathbf{P}\,\theta & (\theta \text{ occurred in the Past}) \\
& & \mid \ \mathbf{H}\,\theta & (\theta \text{ has always occurred (Historically)}) \\
& & \mid \ \theta_1 \,\mathbf{S}\, \theta_2 & (\theta_1 \text{ has held Since } \theta_2)
\end{array}
$$

For example,

$$\mathbf{G}\,(\mathsf{PEDAL} \rightarrow \mathbf{F}\,\mathsf{BRAKE})$$

$$\mathbf{G}\,(\mathsf{BRAKE} \rightarrow \mathbf{P}\,\mathsf{PEDAL})$$

Temporal logic: LTL

Linear Temporal Logic (LTL): Propositional logic with temporal modalities:

$$
\begin{array}{rcll}
\theta & ::= & P \mid \theta_1 \wedge \theta_2 \mid \theta_1 \vee \theta_2 \mid \neg\theta & \\
 & & \mid \mathbf{F}\,\theta & (\theta \text{ occurs in the Future}) \\
 & & \mid \mathbf{G}\,\theta & (\theta \text{ occurs always (Globally)}) \\
 & & \mid \theta_1\ \mathbf{U}\ \theta_2 & (\theta_1 \text{ holds Until } \theta_2) \\
 & & \mid \mathbf{P}\,\theta & (\theta \text{ occurred in the Past}) \\
 & & \mid \mathbf{H}\,\theta & (\theta \text{ has always occurred (Historically)}) \\
 & & \mid \theta_1\ \mathbf{S}\ \theta_2 & (\theta_1 \text{ has held Since } \theta_2)
\end{array}
$$

For example,

$$\mathbf{G}\,(\mathsf{PEDAL} \rightarrow \mathbf{F}\,\mathsf{BRAKE})$$

$$\mathbf{G}\,(\mathsf{BRAKE} \rightarrow \mathbf{P}\,\mathsf{PEDAL})$$

LTL is all you need

LTL has emerged as *the* definitive temporal logic in the classical setting.

Theorem (Kamp 1968, GPSS 1980)

LTL is as expressive as FO($<$).

LTL is all you need

LTL has emerged as *the* definitive temporal logic in the classical setting.

Theorem (Kamp 1968, GPSS 1980)

LTL is as expressive as FO($<$).

Classic temporal logic

Metric temporal logic

Extending Kamp’s Theorem (again)

Quantitative setting

In reality, timed systems are usually quantitative.

LTL can express:

"If I press the brake pedal then the pads will be applied."

Quantitative setting

In reality, timed systems are usually quantitative.

LTL can express:

“*If I press the brake pedal then the pads will be applied.*”

Quantitative setting

In reality, timed systems are usually quantitative.

Want to specify:

"*If I press the brake pedal then*
the pads will be applied between 0.5ms and 1ms."

Adding time metrics to the models

We want to add a metric to the model so we can enforce certain timing constraints, for example:

“Apply brake pads between 5 *to* 10 *time units after pedal is pushed”*.

$\mathbb{R}$ has a distance metric $\rightsquigarrow$ use real intervals for timing constraints.

- Traditional approach: intervals over $\mathbb{Z}$.
- Continuous but finitely presentable: intervals over $\mathbb{Q}$ (seen in Part I).
- **NEW!** Intervals over an arbitrary additive subgroup of $\mathbb{R}$...

Adding time metrics to the models

We want to add a metric to the model so we can enforce certain timing constraints, for example:

"Apply brake pads between 5 *to* 10 *time units after pedal is pushed"*.

$\mathbb{R}$ has a distance metric $\rightsquigarrow$ use real intervals for timing constraints.

- Traditional approach: intervals over $\mathbb{Z}$.
- Continuous but finitely presentable: intervals over $\mathbb{Q}$ (seen in Part I).
- **NEW!** Intervals over an arbitrary additive subgroup of $\mathbb{R}$...

Adding time metrics to the models

We want to add a metric to the model so we can enforce certain timing constraints, for example:

"Apply brake pads between 5 *to* 10 *time units after pedal is pushed"*.

$\mathbb{R}$ has a distance metric $\rightsquigarrow$ use real intervals for timing constraints.

- Traditional approach: intervals over $\mathbb{Z}$.
- Continuous but finitely presentable: intervals over $\mathbb{Q}$ (seen in Part I).
- **NEW!** Intervals over an arbitrary additive subgroup of $\mathbb{R}$...

Adding time metrics to the models

We want to add a metric to the model so we can enforce certain timing constraints, for example:

"Apply brake pads between 5 *to* 10 *time units after pedal is pushed"*.

$\mathbb{R}$ has a distance metric $\rightsquigarrow$ use real intervals for timing constraints.

- Traditional approach: intervals over $\mathbb{Z}$.
- Continuous but finitely presentable: intervals over $\mathbb{Q}$ (seen in Part I).
- **NEW!** Intervals over an arbitrary additive subgroup of $\mathbb{R}$...

Adding time metrics to the models

We want to add a metric to the model so we can enforce certain timing constraints, for example:

"Apply brake pads between 5 *to* 10 *time units after pedal is pushed"*.

$\mathbb{R}$ has a distance metric $\rightsquigarrow$ use real intervals for timing constraints.

- Traditional approach: intervals over $\mathbb{Z}$.
- Continuous but finitely presentable: intervals over $\mathbb{Q}$ (seen in Part I).
- **NEW!** Intervals over an arbitrary additive subgroup of $\mathbb{R}$...

Additive subgroup?

- Can easily form integer linear combinations of timing constants.
- Integer linear combinations of $\mathcal{K}$ = Subgroup of $(\mathbb{R}, +)$ generated by $\mathcal{K}$.

Motivation

- Includes most general case ($\mathcal{K} = \mathbb{R}$)
- Generalizes previous cases ($\mathcal{K} = \mathbb{Z},\ \mathbb{Q},$ or $\{0\}$)
- Can be used to model multiple independent asynchronous timing systems (e.g. $\mathbb{Z}[\sqrt{2}]$)

“Turn on brake lights 5 ‘pedal-time-units’ after pedal is pressed and 3 ‘pad-time-units’ after pads are applied”

Motivation

- Includes most general case ($\mathcal{K} = \mathbb{R}$)
- Generalizes previous cases ($\mathcal{K} = \mathbb{Z},\ \mathbb{Q},$ or $\{0\}$)
- Can be used to model multiple independent asynchronous timing systems (e.g. $\mathbb{Z}[\sqrt{2}]$)

"Turn on brake lights 5 'pedal-time-units' after pedal is pressed and 3 'pad-time-units' after pads are applied"

Motivation

- Includes most general case ($\mathcal{K} = \mathbb{R}$)
- Generalizes previous cases ($\mathcal{K} = \mathbb{Z},\ \mathbb{Q},$ or $\{0\}$)
- Can be used to model multiple independent asynchronous timing systems (e.g. $\mathbb{Z}[\sqrt{2}]$)

“Turn on brake lights 5 *‘pedal-time-units’ after pedal is pressed and* 3 *‘pad-time-units’ after pads are applied”*

Metric Predicate Logic

We add many unary functions $\{+c : c \in \mathcal{K}\}$ to FO($<$) to model moving c time units into the future.

"Apply brake pads between 5 *to* 10 *time units after pedal is pushed"*

becomes

$$\forall x.\mathrm{PEDAL}(x) \to \exists y.\,(x+5 < y < x+10) \wedge \mathrm{BRAKE}(y),$$

a formula of $\mathrm{FO}_{\{5,10\}}$.

Metric Predicate Logic

We add many unary functions $\{+c : c \in \mathcal{K}\}$ to FO($<$) to model moving c time units into the future.

"Apply brake pads between 5 *to* 10 *time units after pedal is pushed"*

becomes

$$\forall x.\mathrm{PEDAL}(x) \rightarrow \exists y.\,(x{+}5 < y < x{+}10) \wedge \mathrm{BRAKE}(y),$$

a formula of $\mathrm{FO}_{\{5,10\}}$.

Metric Temporal Logic

Metric Temporal Logic (MTL) [Koymans; de Roever; Pnueli ∼1990] is a central quantitative specification formalism for timed systems.

$\text{MTL}_{\mathcal{K}}$ = LTL + timing constraints on operators:

$$\mathbf{G}\,(\text{PEDAL} \rightarrow \mathbf{F}_{(5,10)}\ \text{BRAKE})$$

Formally,

$$\begin{array}{rcll}
\theta & ::= & P \mid \theta_1 \wedge \theta_2 \mid \theta_1 \vee \theta_2 \mid \neg\theta & \\
 & & \mid \mathbf{F}_I\, \theta & (\theta \text{ occurs in the Future in the interval } I) \\
 & & \mid \mathbf{G}_I\, \theta & (\theta \text{ occurs always (Globally) in the interval } I) \\
 & & \mid \theta_1\, \mathbf{U}_I\, \theta_2 & (\theta_2 \text{ holds in } I \text{ and Until then } \theta_1 \text{ holds}) \\
 & & \mid \mathbf{P}_I\, \theta & \theta \text{ occurred in the Past in the interval } I) \\
 & & \mid \mathbf{H}_I\, \theta & (\theta \text{ always occurred in the interval } I) \\
 & & \mid \theta_1\, \mathbf{S}_I\, \theta_2 & (\ \theta_2 \text{ held in } I \text{ and } \theta_1 \text{ has held Since})
\end{array}$$

where I is an interval of $\mathcal{K}$.

Metric Temporal Logic

Metric Temporal Logic (MTL) [Koymans; de Roever; Pnueli ∼1990] is a central quantitative specification formalism for timed systems.

MTL$_{\mathcal{K}}$ = LTL + timing constraints on operators:

$$\mathbf{G}\,(\text{PEDAL} \rightarrow \mathbf{F}_{(5,10)}\ \text{BRAKE})$$

Formally,

$$
\begin{array}{lllr}
\theta & ::= & P \mid \theta_1 \wedge \theta_2 \mid \theta_1 \vee \theta_2 \mid \neg\theta & \\
& & \mid \mathbf{F}_I\,\theta & (\theta \text{ occurs in the Future in the interval } I) \\
& & \mid \mathbf{G}_I\,\theta & (\theta \text{ occurs always (Globally) in the interval } I) \\
& & \mid \theta_1\,\mathbf{U}_I\,\theta_2 & (\theta_2 \text{ holds in } I \text{ and Until then } \theta_1 \text{ holds}) \\
& & \mid \mathbf{P}_I\,\theta & \theta \text{ occurred in the Past in the interval } I) \\
& & \mid \mathbf{H}_I\,\theta & (\theta \text{ always occurred in the interval } I) \\
& & \mid \theta_1\,\mathbf{S}_I\,\theta_2 & (\ \theta_2 \text{ held in } I \text{ and } \theta_1 \text{ has held Since})
\end{array}
$$

where I is an interval of $\mathcal{K}$.

Metric Temporal Logic

Metric Temporal Logic (MTL) [Koymans; de Roever; Pnueli ∼1990] is a central quantitative specification formalism for timed systems.

$\text{MTL}_{\mathcal{K}}$ = LTL + timing constraints taken from $\mathcal{K}$ on operators:

$$\mathbf{G}\,(\text{PEDAL} \rightarrow \mathbf{F}_{(5,10)}\ \text{BRAKE})$$

Formally,

$$\begin{aligned}
\theta \quad ::= \quad & P \mid \theta_1 \wedge \theta_2 \mid \theta_1 \vee \theta_2 \mid \neg\theta \\
& \mid \mathbf{F}_I\,\theta && (\theta \text{ occurs in the Future in the interval } I) \\
& \mid \mathbf{G}_I\,\theta && (\theta \text{ occurs always (Globally) in the interval } I) \\
& \mid \theta_1\,\mathbf{U}_I\,\theta_2 && (\theta_2 \text{ holds in } I \text{ and Until then } \theta_1 \text{ holds}) \\
& \mid \mathbf{P}_I\,\theta && \theta \text{ occurred in the Past in the interval } I) \\
& \mid \mathbf{H}_I\,\theta && (\theta \text{ always occurred in the interval } I) \\
& \mid \theta_1\,\mathbf{S}_I\,\theta_2 && (\ \theta_2 \text{ held in } I \text{ and } \theta_1 \text{ has held Since})
\end{aligned}$$

where I is an interval of $\mathcal{K}$.

Metric Temporal Logic

Metric Temporal Logic (MTL) [Koymans; de Roever; Pnueli ∼1990] is a central quantitative specification formalism for timed systems.

$\mathsf{MTL}_{\mathcal{K}}$ = LTL + timing constraints taken from $\mathcal{K}$ on operators:

$$\mathbf{G}\,(\mathsf{PEDAL} \rightarrow \mathbf{F}_{(5,10)}\,\mathsf{BRAKE})$$

Formally,

$$
\begin{array}{rcll}
\theta & ::= & P \mid \theta_1 \wedge \theta_2 \mid \theta_1 \vee \theta_2 \mid \neg\theta & \\
 & & \mid \mathbf{F}_I\,\theta & (\theta \text{ occurs in the Future in the interval } I) \\
 & & \mid \mathbf{G}_I\,\theta & (\theta \text{ occurs always (Globally) in the interval } I) \\
 & & \mid \theta_1\,\mathbf{U}_I\,\theta_2 & (\theta_2 \text{ holds in } I \text{ and Until then } \theta_1 \text{ holds}) \\
 & & \mid \mathbf{P}_I\,\theta & \theta \text{ occurred in the Past in the interval } I) \\
 & & \mid \mathbf{H}_I\,\theta & (\theta \text{ always occurred in the interval } I) \\
 & & \mid \theta_1\,\mathbf{S}_I\,\theta_2 & (\,\theta_2 \text{ held in } I \text{ and } \theta_1 \text{ has held Since})
\end{array}
$$

where I is an interval of $\mathcal{K}$.

Metric Temporal Logic

Metric Temporal Logic (MTL) [Koymans; de Roever; Pnueli $\sim$1990] is a central quantitative specification formalism for timed systems.

$\mathrm{MTL}_{\mathcal{K}}$ = LTL + timing constraints taken from $\mathcal{K}$ on operators:

$$\mathbf{G}\,(\mathrm{PEDAL} \rightarrow \mathbf{F}_{(5,10)}\,\mathrm{BRAKE})$$

Formally,

$$\begin{array}{rcll} \theta & ::= & P \mid \theta_1 \wedge \theta_2 \mid \theta_1 \vee \theta_2 \mid \neg\theta & \\ & & \mid \mathbf{F}_I\,\theta & (\theta \text{ occurs in the Future in the interval } I) \\ & & \mid \mathbf{G}_I\,\theta & (\theta \text{ occurs always (Globally) in the interval } I) \\ & & \mid \theta_1\,\mathbf{U}_I\,\theta_2 & (\theta_2 \text{ holds in } I \text{ and Until then } \theta_1 \text{ holds}) \\ & & \mid \mathbf{P}_I\,\theta & \theta \text{ occurred in the Past in the interval } I) \\ & & \mid \mathbf{H}_I\,\theta & (\theta \text{ always occurred in the interval } I) \\ & & \mid \theta_1\,\mathbf{S}_I\,\theta_2 & (\ \theta_2 \text{ held in } I \text{ and } \theta_1 \text{ has held Since}) \end{array}$$

where I is an interval of $\mathcal{K}$.

Classic temporal logic

Metric temporal logic

Extending Kamp’s Theorem (again)

Kamp's theorem restated

Theorem (Kamp 1968)

$MTL_{\{0\}}$ has the same expressive power as $FO_{\{0\}}$.

Part I recap

Theorem (Hirshfeld & Rabinovich 2007)

$MTL_{\mathbb{Z}}$ is strictly less expressive than $FO_{\mathbb{Z}}$.

Part I recap

Theorem (H., Ouaknine & Worrell 2013)

$MTL_{\mathbb{Q}}$ has the same expressive power as $FO_{\mathbb{Q}}$.

What about $MTL_{\mathbb{R}}$? or $MTL_{\mathbb{Z}[\sqrt{2}]}$?

A true extension of Kamp's theorem

Theorem (H. 2013)

$MTL_{\mathcal{K}} = FO_{\mathcal{K}}$ if and only if $\mathcal{K}$ is dense.

Proof: “Only if”

Lemma

If $\mathcal{K}$ is a non-dense additive subgroup of $\mathbb{R}$ then $\mathcal{K} = \epsilon\mathbb{Z}$ for some $\epsilon \in \mathbb{R}$.

Proof: “If”

Recall proof in Part I:

1. Use metric separation to reduce to bounded formulas.
2. Scale $FO_{\mathbb{Q}}$ formula to get a formula in $FO_{\mathbb{Z}}$.
3. Use “stacking” to remove the $+1$ function.
4. Use denseness of $\mathbb{Q}$ to express LTL statements restricted to a single time interval.
5. Scale to remove the factor introduced in Step 2.

Proof: “If”

Recall proof in Part I:

1. Use metric separation to reduce to bounded formulas.
2. Scale $FO_{\mathbb{Q}}$ formula to get a formula in $FO_{\mathbb{Z}}$.
3. Use “stacking” to remove the $+1$ function.
4. Use denseness of $\mathbb{Q}$ to express LTL statements restricted to a single time interval.
5. Scale to remove the factor introduced in Step 2.

Proof: “If”

Recall proof in Part I:

1. Use metric separation to reduce to bounded formulas.
2. Scale $FO_{\mathbb{Q}}$ formula to get a formula in $FO_{\mathbb{Z}}$.
3. Use “stacking” to remove the $+1$ function.
4. Use denseness of $\mathbb{Q}$ to express LTL statements restricted to a single time interval.
5. Scale to remove the factor introduced in Step 2.

Proof: “If”

Recall proof in Part I:

1. Use metric separation to reduce to bounded formulas.
2. Scale $FO_{\mathbb{Q}}$ formula to get a formula in $FO_{\mathbb{Z}}$.
3. Use “stacking” to remove the $+1$ function.
4. Use denseness of $\mathbb{Q}$ to express LTL statements restricted to a single time interval.
5. Scale to remove the factor introduced in Step 2.

Proof: “If”

Recall proof in Part I:

1. Use metric separation to reduce to bounded formulas.
2. Scale $\mathrm{FO}_{\mathbb{Q}}$ formula to get a formula in $\mathrm{FO}_{\mathbb{Z}}$.
3. Use “stacking” to remove the $+1$ function.
4. Use denseness of $\mathbb{Q}$ to express LTL statements restricted to a single time interval.
5. Scale to remove the factor introduced in Step 2.

Proof: “If”

Recall proof in Part I:

1. Use metric separation to reduce to bounded formulas.
2. ~~Scale $FO_{\mathbb{Q}}$ formula to get a formula in $FO_{\mathbb{Z}}$~~.
3. ~~Use “stacking” to remove the $+1$ function~~.
4. Use denseness of $\mathbb{Q}$ to express LTL statements restricted to a single time interval.
5. ~~Scale to remove the factor introduced in Step 2~~.

Proof: “If”

1. Use more general metric separation to reduce to bounded formulas.
2. Use a normal form for $FO_{\mathcal{K}}$ formulas to remove $+c$ functions.
3. Use denseness of $\mathcal{K}$ to express LTL statements restricted to an interval.

Proof: “If”

1. Use more general metric separation to reduce to bounded formulas.
2. Use a normal form for $\mathrm{FO}_{\mathcal{K}}$ formulas to remove $+c$ functions.
3. Use denseness of $\mathcal{K}$ to express LTL statements restricted to an interval.

Proof: “If”

1. Use more general metric separation to reduce to bounded formulas.
2. Use a normal form for $\mathrm{FO}_{\mathcal{K}}$ formulas to remove $+c$ functions.
3. Use denseness of $\mathcal{K}$ to express LTL statements restricted to an interval.

Proof: “If”

1. Use more general metric separation to reduce to bounded formulas.
2. Use a normal form for $FO_{\mathcal{K}}$ formulas to remove $+c$ functions.
3. Use denseness of $\mathcal{K}$ to express LTL statements restricted to an interval.

Separation of temporal logics

A temporal logic formula is

- pure past if it is invariant on flows that agree on the past
- pure present if is invariant on flows that agree on the present
- pure future if is invariant on flows that agree on the future

A temporal logic is separable if all its formulas are equivalent to a boolean combination of pure past, present and future formulas.

Separation of temporal logics

A temporal logic formula is

- pure past if it is invariant on flows that agree on the past
- pure present if is invariant on flows that agree on the present
- pure future if is invariant on flows that agree on the future

A temporal logic is separable if all its formulas are equivalent to a boolean combination of pure past, present and future formulas.

For example,

$$\mathbf{G}(\mathsf{BRAKE} \rightarrow \mathbf{P}\,\mathsf{PEDAL})$$
$$= \mathbf{P}\,\mathsf{PEDAL} \;\vee\; (\neg\mathsf{BRAKE}\;\mathbf{U}\;\mathsf{PEDAL}) \;\vee\; \mathbf{G}(\neg\mathsf{BRAKE})$$

Separation of temporal logics

A temporal logic formula is

- pure past if it is invariant on flows that agree on the past
- pure present if is invariant on flows that agree on the present
- pure future if is invariant on flows that agree on the future

A temporal logic is separable if all its formulas are equivalent to a boolean combination of pure past, present and future formulas.

For example,

$$\mathbf{G}(\mathsf{BRAKE} \rightarrow \mathbf{P}\,\mathsf{PEDAL})$$
$$= \mathbf{P}\,\mathsf{PEDAL} \;\vee\; (\neg\mathsf{BRAKE}\,\mathbf{U}\,\mathsf{PEDAL}) \;\vee\; \mathbf{G}(\neg\mathsf{BRAKE})$$

Separation of temporal logics

A temporal logic formula is

- pure past if it is invariant on flows that agree on the past
- pure present if is invariant on flows that agree on the present
- pure future if is invariant on flows that agree on the future

A temporal logic is separable if all its formulas are equivalent to a boolean combination of pure past, present and future formulas.

Lemma
LTL is separable.

Gabbay's theorem

Theorem (Gabbay 1981)

A temporal logic is expressively complete if and only if it is separable.

Quantitative separation

Separation does not hold in the quantitative setting.

For example,

$$\mathbf{G}(\mathtt{BRAKE} \rightarrow \mathbf{P}_{(5,10)}\mathtt{PEDAL})$$

General quantitative separation

Given a constant $c > 0$, a metric temporal formula is:

- pure c-distant past if it is invariant on flows that agree on $(-\infty, -c)$
- pure c-distant future if it is invariant on flows that agree on (c, ∞)
- bounded if there is an N such that it is invariant on all flows that agree on $(-N, N)$

A temporal logic with constants from $\mathcal{K}$ is generally metrically separable if every formula is equivalent, for some $c \in \mathcal{K}_{>0}$, to a boolean combination of pure c-distant past, pure c-distant future and bounded formulas.

Lemma

$MTL_{\mathcal{K}}$ is generally metrically separable for non-trivial $\mathcal{K}$.

General quantitative separation

Given a constant $c > 0$, a metric temporal formula is:

- pure c-distant past if it is invariant on flows that agree on $(-\infty, -c)$
- pure c-distant future if it is invariant on flows that agree on (c, ∞)
- bounded if there is an N such that it is invariant on all flows that agree on $(-N, N)$

A temporal logic with constants from $\mathcal{K}$ is generally metrically separable if every formula is equivalent, for some $c \in \mathcal{K}_{>0}$, to a boolean combination of pure c-distant past, pure c-distant future and bounded formulas.

Lemma

$MTL_{\mathcal{K}}$ is generally metrically separable for non-trivial $\mathcal{K}$.

General quantitative separation

Given a constant $c > 0$, a metric temporal formula is:

- pure c-distant past if it is invariant on flows that agree on $(-\infty, -c)$
- pure c-distant future if it is invariant on flows that agree on (c, ∞)
- bounded if there is an N such that it is invariant on all flows that agree on $(-N, N)$

A temporal logic with constants from $\mathcal{K}$ is generally metrically separable if every formula is equivalent, for some $c \in \mathcal{K}_{>0}$, to a boolean combination of pure c-distant past, pure c-distant future and bounded formulas.

Corollary

$MTL_{\mathcal{K}} = FO_{\mathcal{K}}$ iff $MTL_{\mathcal{K}}$ can express all bounded $FO_{\mathcal{K}}$ formulas.

Proof: "If"

1. Use more general metric separation to reduce to bounded formulas.
2. Use a normal form for $\mathrm{FO}_{\mathcal{K}}$ formulas to remove $+c$ functions.
3. Use denseness of $\mathcal{K}$ to express LTL statements restricted to an interval.

Removing the unary functions

First move the unary functions to the free variable (removing from predicates as before).

$$\begin{aligned}
\varphi(x) &= \exists y \in (x, x+1)\ \exists z \in (y, y+\sqrt{2}) \ldots \\
&= \exists y \in (x, x+1)\ \big(\exists z \in (y, x+1) \ldots \\
&\qquad\qquad \vee\ \exists z \in (x+1, y+\sqrt{2}) \ldots \big) \\
&= \exists y \in (x, x+1)\ \big(\exists z \in (y, x+1) \ldots \\
&\qquad\qquad \vee\ \exists z' \in (x+1-\sqrt{2}, y) \ldots \big) \\
\varphi(x_0, x_1, x_2) &= \exists y \in (x_1, x_2)\ \big(\exists z \in (y, x_2) \ldots \vee \exists z' \in (x_0, y) \ldots \big)
\end{aligned}$$

Corollary

$MTL_{\mathcal{K}} = FO_{\mathcal{K}}$ iff $MTL_{\mathcal{K}}$ can express all bounded $FO_{\{0\}}$ formulas.

Removing the unary functions

First move the unary functions to the free variable (removing from predicates as before).

$$\varphi(x) = \exists y \in (x, x+1)\ \exists z \in (y, y+\sqrt{2}) \ldots$$

$$= \exists y \in (x, x+1)\ \big(\exists z \in (y, x+1) \ldots \vee \exists z \in (x+1, y+\sqrt{2}) \ldots\big)$$

$$= \exists y \in (x, x+1)\ \big(\exists z \in (y, x+1) \ldots \vee \exists z' \in (x+1-\sqrt{2}, y) \ldots\big)$$

$$\varphi(x_0, x_1, x_2) = \exists y \in (x_1, x_2)\ \big(\exists z \in (y, x_2) \ldots \vee \exists z' \in (x_0, y) \ldots\big)$$

Corollary

$MTL_{\mathcal{K}}$ = $FO_{\mathcal{K}}$ iff $MTL_{\mathcal{K}}$ can express all bounded $FO_{\{0\}}$ formulas.

Removing the unary functions

First move the unary functions to the free variable (removing from predicates as before).

$$
\begin{aligned}
\varphi(x) &= \exists y \in (x, x+1)\ \exists z \in (y, y+\sqrt{2}) \ldots \\
&= \exists y \in (x, x+1)\ \big(\exists z \in (y, x+1) \ldots \\
&\qquad \vee \exists z \in (x+1, y+\sqrt{2}) \ldots \big) \\
&= \exists y \in (x, x+1)\ \big(\exists z \in (y, x+1) \ldots \\
&\qquad \vee \exists z' \in (x+1-\sqrt{2}, y) \ldots \big)
\end{aligned}
$$

$$
\varphi(x_0, x_1, x_2) = \exists y \in (x_1, x_2)\ \big(\exists z \in (y, x_2) \ldots \vee \exists z' \in (x_0, y) \ldots \big)
$$

Corollary

$MTL_{\mathcal{K}} = FO_{\mathcal{K}}$ iff $MTL_{\mathcal{K}}$ can express all bounded $FO_{\{0\}}$ formulas.

Removing the unary functions

First move the unary functions to the free variable (removing from predicates as before).

$$
\begin{aligned}
\varphi(x) &= \exists y \in (x, x+1)\ \exists z \in (y, y+\sqrt{2}) \ldots \\
&= \exists y \in (x, x+1)\ \big(\exists z \in (y, x+1) \ldots \\
&\qquad\qquad \vee\ \exists z \in (x+1, y+\sqrt{2}) \ldots \big) \\
&= \exists y \in (x, x+1)\ \big(\exists z \in (y, x+1) \ldots \\
&\qquad\qquad \vee\ \exists z' \in (x+1-\sqrt{2}, y) \ldots \big)
\end{aligned}
$$

$$
\varphi(x_0, x_1, x_2) = \exists y \in (x_1, x_2)\ \big(\exists z \in (y, x_2) \ldots \vee \exists z' \in (x_0, y) \ldots \big)
$$

Corollary

$MTL_{\mathcal{K}}$ = $FO_{\mathcal{K}}$ iff $MTL_{\mathcal{K}}$ can express all bounded $FO_{\{0\}}$ formulas.

Removing the unary functions

Replace the “milestones” ($\{x+1-\sqrt{2}, x, x+1\}$) with new variables to obtain a FO($<$) formula.

$$\begin{aligned}
\varphi(x) &= \exists y \in (x, x+1)\, \exists z \in (y, y+\sqrt{2}) \ldots \\
&= \exists y \in (x, x+1)\, \big(\exists z \in (y, x+1) \ldots \\
&\qquad \vee \exists z \in (x+1, y+\sqrt{2}) \ldots \big) \\
&= \exists y \in (x, x+1)\, \big(\exists z \in (y, x+1) \ldots \\
&\qquad \vee \exists z' \in (x+1-\sqrt{2}, y) \ldots \big) \\
\varphi(x_0, x_1, x_2) &= \exists y \in (x_1, x_2)\, \big(\exists z \in (y, x_2) \ldots \vee \exists z' \in (x_0, y) \ldots \big)
\end{aligned}$$

Corollary

$MTL_{\mathcal{K}}$ = $FO_{\mathcal{K}}$ iff $MTL_{\mathcal{K}}$ can express all bounded $FO_{\{0\}}$ formulas.

Removing the unary functions

Use a model-theoretic argument to break this into formulas on the intervals $\{x_0\}, (x_0, x_1), \{x_1\}, \ldots$

$$
\begin{aligned}
\varphi(x) &= \exists y \in (x, x+1)\ \exists z \in (y, y+\sqrt{2}) \ldots \\
&= \exists y \in (x, x+1)\ \big(\exists z \in (y, x+1) \ldots \\
&\qquad\qquad \vee\ \exists z \in (x+1, y+\sqrt{2}) \ldots\big) \\
&= \exists y \in (x, x+1)\ \big(\exists z \in (y, x+1) \ldots \\
&\qquad\qquad \vee\ \exists z' \in (x+1-\sqrt{2}, y) \ldots\big) \\
\varphi(x_0, x_1, x_2) &= \exists y \in (x_1, x_2)\ \big(\exists z \in (y, x_2) \ldots \vee \exists z' \in (x_0, y) \ldots\big)
\end{aligned}
$$

Corollary

$MTL_{\mathcal{K}} = FO_{\mathcal{K}}$ iff $MTL_{\mathcal{K}}$ can express all bounded $FO_{\{0\}}$ formulas.

Removing the unary functions

Use a model-theoretic argument to break this into formulas on the intervals $\{x_0\}, (x_0, x_1), \{x_1\}, \ldots$

$$
\begin{aligned}
\varphi(x) &= \exists y \in (x, x+1)\, \exists z \in (y, y+\sqrt{2}) \ldots \\
&= \exists y \in (x, x+1)\, \big(\exists z \in (y, x+1) \ldots \\
&\qquad\qquad \vee\, \exists z \in (x+1, y+\sqrt{2}) \ldots \big) \\
&= \exists y \in (x, x+1)\, \big(\exists z \in (y, x+1) \ldots \\
&\qquad\qquad \vee\, \exists z' \in (x+1-\sqrt{2}, y) \ldots \big) \\
\varphi(x_0, x_1, x_2) &= \exists y \in (x_1, x_2)\, \big(\exists z \in (y, x_2) \ldots \vee \exists z' \in (x_0, y) \ldots \big)
\end{aligned}
$$

Corollary

$MTL_{\mathcal{K}}$ = $FO_{\mathcal{K}}$ iff $MTL_{\mathcal{K}}$ can express all bounded $FO_{\{0\}}$ formulas.

Proof: “If”

1. Use more general metric separation to reduce to bounded formulas.
2. Use a normal form for $\mathrm{FO}_{\mathcal{K}}$ formulas to remove $+c$ functions.
3. Use denseness of $\mathcal{K}$ to express LTL statements restricted to an interval.

Failure of Kamp's theorem

$\mathrm{MTL}_{\mathbb{Z}}$ is unable to express:

"*P occurs twice in the next time interval.*"

In $\mathrm{FO}_{\mathbb{Z}}$:

$$\varphi(z) = \exists x . \exists y . (z < x < z + \ ) \wedge (z < y < z + \ ) \wedge \mathrm{P}(x) \wedge \mathrm{P}(y).$$

In $\mathrm{MTL}_{\mathbb{Z}}$:

$$\left(\mathbf{F}_{(0,1)}\mathrm{P} \wedge \mathbf{F}_{(1,2)}\mathrm{P}\right) \quad \vee$$
$$\mathbf{F}_{=2}\left(\mathbf{P}_{(0,1)}\left(\mathrm{P} \wedge \mathbf{P}_{(0,1)}\mathrm{P}\right)\right)$$

Failure of Kamp's theorem

$\text{MTL}_{\mathbb{Z}}$ is unable to express:

"*P occurs twice in the next time interval.*"

In $\text{FO}_{\mathbb{Z}}$:

$$\varphi(z) = \exists x.\exists y.(z < x < z+1) \wedge (z < y < z+1) \wedge \text{P}(x) \wedge \text{P}(y).$$

In $\text{MTL}_{\mathbb{Z}}$:

???

$$(\mathbf{F}_{(0,1)}\text{P} \wedge \mathbf{F}_{(1,2)}\text{P}) \quad \vee$$

$$\mathbf{F}_{=2}\Big(\mathbf{P}_{(0,1)}\big(\text{P} \wedge \mathbf{P}_{(0,1)}\text{P}\big)\Big)$$

Failure of Kamp's theorem

$\mathrm{MTL}_{\mathbb{Z}}$ is able to express:

"*P occurs twice in the next two time intervals.*"

In $\mathrm{FO}_{\mathbb{Z}}$:

$$\varphi(z) = \exists x.\exists y.(z < x < z + 2) \wedge (z < y < z + 2) \wedge \mathrm{P}(x) \wedge \mathrm{P}(y).$$

In $\mathrm{MTL}_{\mathbb{Z}}$:

$$\begin{aligned} \varphi \quad = \quad & \mathbf{F}_{(0,1)}\big(\mathrm{P} \wedge \mathbf{F}_{(0,1)}\mathrm{P}\big) \quad \vee \\ & \big(\mathbf{F}_{(0,1)}\mathrm{P} \wedge \mathbf{F}_{(1,2)}\mathrm{P}\big) \quad \vee \\ & \mathbf{F}_{=2}\Big(\mathbf{P}_{(0,1)}\big(\mathrm{P} \wedge \mathbf{P}_{(0,1)}\mathrm{P}\big)\Big) \end{aligned}$$

Failure of Kamp's theorem

MTL$_{\mathbb{Z}}$ is able to express:

"*P occurs twice in the next two time intervals.*"

In FO$_{\mathbb{Z}}$:

$$\varphi(z) = \exists x.\exists y.(z < x < z+2) \wedge (z < y < z+2) \wedge \mathsf{P}(x) \wedge \mathsf{P}(y).$$

In MTL$_{\mathbb{Z}}$:

$$\begin{aligned} \varphi \quad = \quad & \mathbf{F}_{(0,1)}\big(\mathsf{P} \wedge \mathbf{F}_{(0,1)}\mathsf{P}\big) \quad \vee \\ & \big(\mathbf{F}_{(0,1)}\mathsf{P} \wedge \mathbf{F}_{(1,2)}\mathsf{P}\big) \quad \vee \\ & \mathbf{F}_{=2}\Big(\mathbf{P}_{(0,1)}\big(\mathsf{P} \wedge \mathbf{P}_{(0,1)}\mathsf{P}\big)\Big) \end{aligned}$$

Failure of Kamp's theorem

$\mathrm{MTL}_{\mathbb{Z}}$ is able to express:

"*P occurs twice in the next two time intervals.*"

In $\mathrm{FO}_{\mathbb{Z}}$:

$$\varphi(z) = \exists x.\exists y.(z < x < z+2) \wedge (z < y < z+2) \wedge \mathrm{P}(x) \wedge \mathrm{P}(y).$$

In $\mathrm{MTL}_{\mathbb{Z}}$:

$$\begin{aligned} \varphi \quad = \quad & \mathbf{F}_{(0,1)}\big(\mathrm{P} \wedge \mathbf{F}_{(0,1)}\mathrm{P}\big) \quad \vee \\ & \big(\mathbf{F}_{(0,1)}\mathrm{P} \wedge \mathbf{F}_{(1,2)}\mathrm{P}\big) \quad \vee \\ & \mathbf{F}_{=2}\Big(\mathbf{P}_{(0,1)}\big(\mathrm{P} \wedge \mathbf{P}_{(0,1)}\mathrm{P}\big)\Big) \end{aligned}$$

Adding granularity

$$\begin{aligned}\varphi \quad = \quad & \mathbf{F}_{(0,1)} \ \big(\mathsf{P} \wedge \mathbf{F}_{(0,1)} \ \mathsf{P}\big) \quad \vee \\ & (\mathbf{F}_{(0,1)} \ \mathsf{P} \wedge \mathbf{F}_{(1,2)} \ \mathsf{P}) \quad \vee \\ & \mathbf{F}_{=2}\Big(\mathbf{P}_{(0,1)} \ \big(\mathsf{P} \wedge \mathbf{P}_{(0,1)} \ \mathsf{P}\big)\Big)\end{aligned}$$

Corollary

"P occurs twice in the next time interval" is expressible in $MTL_{\mathbb{Q}}$.

In fact, $MTL_{\mathcal{K}}$ can express any LTL (and hence $FO_{\{0\}}$) formula "in the next time interval" as long as $\mathcal{K}$ is dense and non-trivial.

Corollary

$MTL_{\mathcal{K}}$ can express $FO_{\mathcal{K}}$ iff $\mathcal{K}$ is dense and non-trivial.

Adding granularity

$$\begin{aligned}\varphi \;=\; & \mathbf{F}_{(0,\frac{1}{2})}\ \big(\mathsf{P} \wedge \mathbf{F}_{(0,\frac{1}{2})}\ \mathsf{P}\big) \quad \vee \\ & \big(\mathbf{F}_{(0,\frac{1}{2})}\ \mathsf{P} \wedge \mathbf{F}_{(\frac{1}{2},1)}\ \mathsf{P}\big) \quad \vee \\ & \mathbf{F}_{=1}\Big(\mathbf{P}_{(0,\frac{1}{2})}\ \big(\mathsf{P} \wedge \mathbf{P}_{(0,\frac{1}{2})}\ \mathsf{P}\big)\Big)\end{aligned}$$

Corollary

"P occurs twice in the next time interval" is expressible in $\mathrm{MTL}_{\mathbb{Q}}$.

In fact, $\mathrm{MTL}_{\mathcal{K}}$ can express any LTL (and hence $\mathrm{FO}_{\{0\}}$) formula "in the next time interval" as long as $\mathcal{K}$ is dense and non-trivial.

Corollary

$\mathrm{MTL}_{\mathcal{K}}$ can express $\mathrm{FO}_{\mathcal{K}}$ iff $\mathcal{K}$ is dense and non-trivial.

Adding granularity

$$\begin{aligned}\varphi \quad = \quad & \mathbf{F}_{(0,\frac{1}{2})}\ \big(\mathsf{P} \wedge \mathbf{F}_{(0,\frac{1}{2})}\ \mathsf{P}\big) \quad \vee \\ & \big(\mathbf{F}_{(0,\frac{1}{2})}\ \mathsf{P} \wedge \mathbf{F}_{(\frac{1}{2},1)}\ \mathsf{P}\big) \quad \vee \\ & \mathbf{F}_{=1}\Big(\mathbf{P}_{(0,\frac{1}{2})}\ \big(\mathsf{P} \wedge \mathbf{P}_{(0,\frac{1}{2})}\ \mathsf{P}\big)\Big)\end{aligned}$$

Corollary

"P occurs twice in the next time interval" is expressible in $MTL_{\mathbb{Q}}$.

In fact, $MTL_{\mathcal{K}}$ can express any LTL (and hence $FO_{\{0\}}$) formula "in the next time interval" as long as $\mathcal{K}$ is dense and non-trivial.

Corollary

$MTL_{\mathcal{K}}$ can express $FO_{\mathcal{K}}$ iff $\mathcal{K}$ is dense and non-trivial.

Adding granularity

$$\begin{aligned}\varphi \quad = \quad & \mathbf{F}_{(0,\frac{1}{2})} \left(\mathsf{P} \wedge \mathbf{F}_{(0,\frac{1}{2})} \mathsf{P}\right) \quad \vee \\ & \left(\mathbf{F}_{(0,\frac{1}{2})} \mathsf{P} \wedge \mathbf{F}_{(\frac{1}{2},1)} \mathsf{P}\right) \quad \vee \\ & \mathbf{F}_{=1}\left(\mathbf{P}_{(0,\frac{1}{2})} \left(\mathsf{P} \wedge \mathbf{P}_{(0,\frac{1}{2})} \mathsf{P}\right)\right)\end{aligned}$$

Corollary

"P occurs twice in the next time interval" is expressible in $MTL_{\mathbb{Q}}$.

In fact, $\text{MTL}_{\mathcal{K}}$ can express any LTL (and hence $\text{FO}_{\{0\}}$) formula "in the next time interval" as long as $\mathcal{K}$ is dense and non-trivial.

Corollary

$MTL_{\mathcal{K}}$ can express $FO_{\mathcal{K}}$ iff $\mathcal{K}$ is dense and non-trivial.

Adding granularity

$$\begin{aligned}\varphi \;=\;& \mathbf{F}_{(0,\frac{1}{2})}\left(\mathsf{P} \wedge \mathbf{F}_{(0,\frac{1}{2})}\ \mathsf{P}\right) \quad \vee \\ & \left(\mathbf{F}_{(0,\frac{1}{2})}\ \mathsf{P} \wedge \mathbf{F}_{(\frac{1}{2},1)}\ \mathsf{P}\right) \quad \vee \\ & \mathbf{F}_{=1}\Big(\mathbf{P}_{(0,\frac{1}{2})}\left(\mathsf{P} \wedge \mathbf{P}_{(0,\frac{1}{2})}\ \mathsf{P}\right)\Big)\end{aligned}$$

Corollary

"P occurs twice in the next time interval" is expressible in MTL$_{\mathbb{Q}}$.

In fact, MTL$_{\mathcal{K}}$ can express any LTL (and hence FO$_{\{0\}}$) formula "in the next time interval" as long as $\mathcal{K}$ is dense and non-trivial.

Corollary

MTL$_{\mathcal{K}}$ can express FO$_{\mathcal{K}}$ iff $\mathcal{K}$ is dense and non-trivial.

A true extension of Kamp's theorem

Theorem

$MTL_{\mathcal{K}}$ = $FO_{\mathcal{K}}$ if and only if $\mathcal{K}$ is dense.

The Expressive Completeness of Metric Temporal Logic II$\frac{1}{2}$:

Count on this

Counting modalities

The counting modalities $\{\mathbf{C}_n : n \in \mathbb{N}\}$ were introduced by Hirshfeld & Rabinovich in 2007.

Intuitively, $\mathbf{C}_n\varphi$ asserts that φ holds in at least n distinct points in the next unit time interval.

MTL with counting is $\mathrm{MTL}_{\mathbb{Z}}$ with the addition of the counting modalities.

A decidability result

Hirshfeld & Rabinovich considered MITL with counting (MTL with counting without singleton intervals).

Theorem (Hirshfeld & Rabinovich 2007)

MITL with counting is decidable.

An expressiveness result

Adding punctuality to MITL with counting gives it the power to express every bounded $FO_{\mathbb{Z}}$ formula, and hence every $FO_{\mathbb{Z}}$ formula.

Theorem (H. 2013)

MTL with counting is expressively equivalent to $FO_{\mathbb{Z}}$.

An expressiveness result

Adding punctuality to MITL with counting gives it the power to express every bounded $FO_{\mathbb{Z}}$ formula, and hence every $FO_{\mathbb{Z}}$ formula.

Theorem (H. 2013)

MTL with counting is expressively equivalent to $FO_{\mathbb{Z}}$.

Conclusions and further work

- Precisely characterized when MTL has the same expressive power as first-order logic.
- Adding counting to the non-equivalent cases gives full expressive power.

Still to do:

- Cost of expressibility.
- Generalization of Gabbay's Theorem.
- Extension to more expressive metric temporal logics.

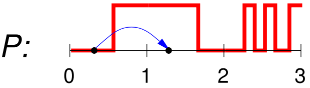
Conclusions and further work

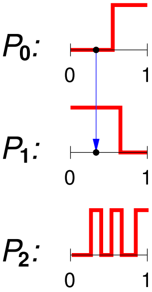
- Precisely characterized when MTL has the same expressive power as first-order logic.
- Adding counting to the non-equivalent cases gives full expressive power.

Still to do:

- Cost of expressibility.
- Generalization of Gabbay's Theorem.
- Extension to more expressive metric temporal logics.

From FO(<,+1) to FO(<)

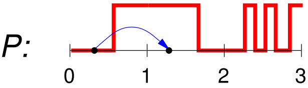


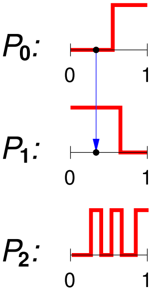


Replace every:

- $\forall x\, \psi(x)$ **by** $\forall x\, (\psi(x) \wedge \psi(x+1) \wedge \psi(x+2))$
- $x + k_1 < y + k_2$ **by** $\begin{cases} x < y & \text{if } k_1 = k_2 \\ \textbf{true} & \text{if } k_1 < k_2 \\ \textbf{false} & \text{if } k_1 > k_2 \end{cases}$
- $P(x+k)$ **by** $P_k(x)$
- After converting to MTL, replace P_k with $\mathbf{F}_{=k}P$

From FO(<,+1) to FO(<)





Replace every:

- $\forall x\, \psi(x)$ **by** $\forall x\ (\psi(x) \wedge \psi(x+1) \wedge \psi(x+2))$
- $x + k_1 < y + k_2$ **by** $\begin{cases} x < y & \text{if } k_1 = k_2 \\ \textbf{true} & \text{if } k_1 < k_2 \\ \textbf{false} & \text{if } k_1 > k_2 \end{cases}$
- $P(x+k)$ **by** $P_k(x)$
- After converting to MTL, replace P_k with $\mathbf{F}_{=k} P$