The Expressiveness of Real-Time Temporal Logics

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(Joint work with Joël Ouaknine and James Worrell)

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Reasoning about time

"Que sera, sera"

Jay Livingston and Ray Evans

- Tense logic introduced by Prior in 1950's
- Used to (automatically) verify reactive and non-terminating systems
  - *"Every* REQ *is followed, at some point, by an* ACK*"*

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- **Linear-time** *vs* **Branching time**
- **Qualitative** (order-theoretic) *vs* **Quantitative** (metric)
- **Expressiveness** *vs* **Computability**

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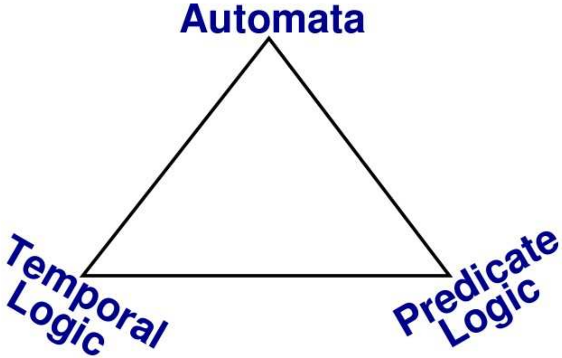

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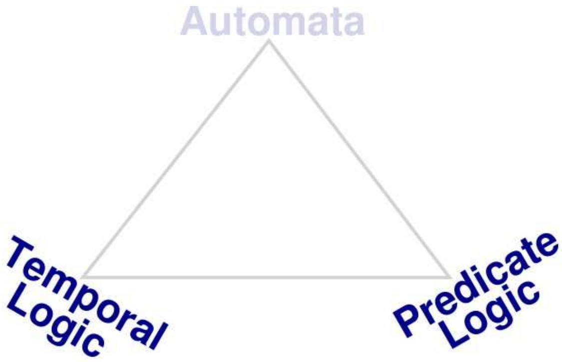

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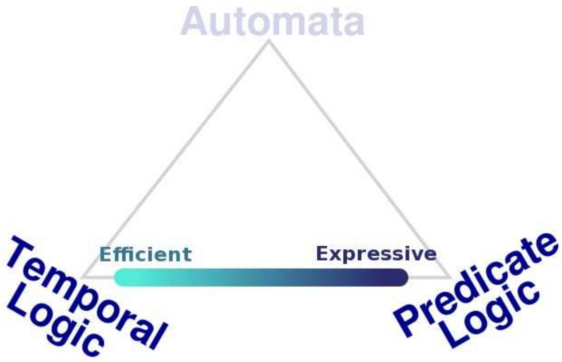


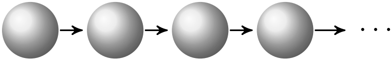
Classic temporal logic

Qualitative Extensions

Quantitative Extensions

Classic temporal models

- A set **MP** of propositions: 
- Discrete time model:



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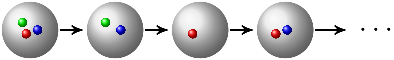

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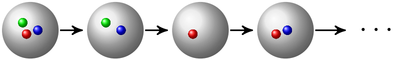

$f : \mathbb{N} \longrightarrow 2^{\mathbf{MP}}$ (flow or signal)

Classic temporal predicate logic

First-order logic (FO($<$)):

$$\varphi ::= x < y \mid P(x) \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi \mid \forall x\, \varphi \mid \exists x \varphi$$

For example: $\forall x\,.\,\mathrm{REQ}(x) \rightarrow \exists y\,.\,\big((y > x) \wedge \mathrm{ACK}(y)\big)$.

Unable to express:

"P happens at every even position (and may or may not hold at odd times)"

Monadic second-order logic (MSO($<$)):

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Büchi's theorem

Flows over non-negative integer time are infinite words over the alphabet $2^{\mathbf{MP}}$.

Theorem (Büchi 1960)

Any MSO($<$) formula φ can be effectively translated into an equivalent automaton A_φ.

Corollary (Church 1960)

The satisfiability problems for MSO($<$) and FO($<$) are decidable.

What about complexity?

Theorem (Stockmeyer 1974)

The satisfiability problem for FO($<$) has non-elementary complexity.

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Linear Temporal Logic (LTL):

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$\mid \mathbf{X}\,\theta$ (θ occurs at the neXt time)

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$\mid \mathbf{G}\,\theta$ (θ occurs always (Globally))

$\mid \theta_1 \,\mathbf{U}\, \theta_2$ (θ_1 holds Until θ_2)

For example, $\mathbf{G}\,(\mathrm{REQ} \rightarrow \mathbf{F}\,\mathrm{ACK})$.

LTL is subsumed by FO($<$), for example

$$P \,\mathbf{U}\, Q \quad \equiv \quad \exists x\, (Q(x) \wedge \forall y\,.\,(y < x) \rightarrow P(y)).$$

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"Every ACK *was preceded, at some point, by a* REQ*"*.

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Is it more expressive?

Kamp's theorem says no.

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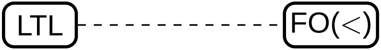
Other temporal logics

Towards MSO($<$):

- ETL: LTL with existential quantification
- μTL: LTL with fix-points
- LDL: LTL with regular expressions
- ...

Expressiveness in the classical case



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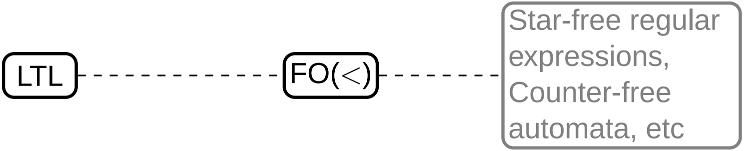

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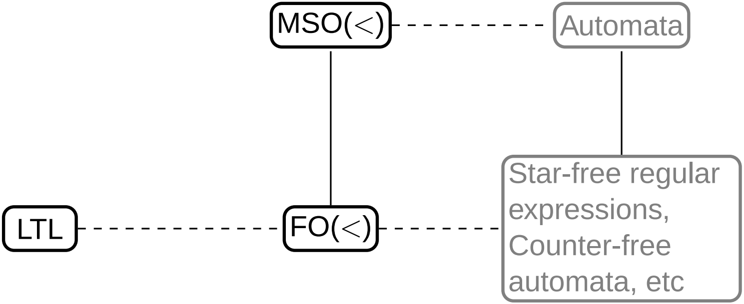

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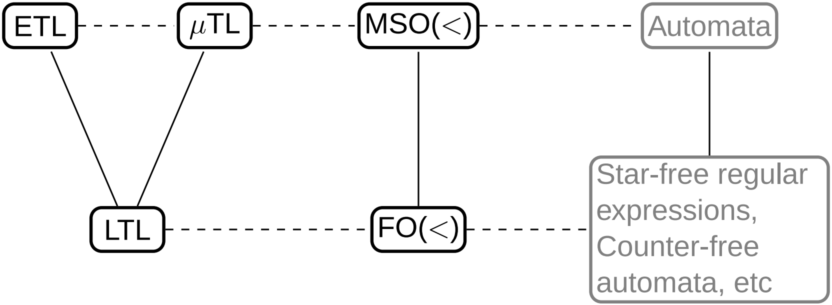

Classic temporal logic

Qualitative Extensions

Quantitative Extensions

Qualitative extensions: Integer time

Non-negative time makes sense in system verification but not necessarily from a philosophical perspective.

Classical:

- Flows: $f : \ \to 2^{\mathbf{MP}}$
- Predicate logics: FO($<$) and MSO($<$)
- Temporal logics: LTL, LTL+Past, ...

Decidability and Expressiveness results (with LTL+Past) still go through.

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Discrete time makes sense in system verification but not necessarily from a physical perspective.

How to handle semantics?

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$f : \mathbb{Q} \rightarrow 2^{\mathbf{MP}}$

$f : \mathbb{R}_{\geq 0} \rightarrow 2^{\mathbf{MP}}$

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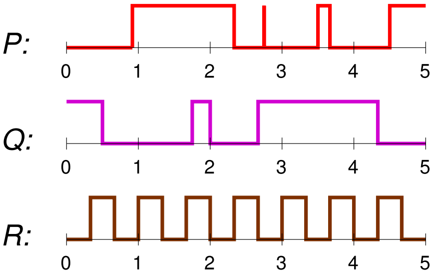

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Pointwise (time-stamped events):

$$(\tau_1, p_1), (\tau_2, p_2), (\tau_3, p_3), \ldots$$

where $\tau_1 < \tau_2 < \ldots \in \mathbb{Q}$ (or $\mathbb{R}$)
and $p_i \in \mathbf{MP}$

Qualitative extensions: Dense time

Some good news:

Theorem (Rabin 1969)

MSO($<$) is decidable over $\mathbb{Q}$.

Some bad news:

Theorem (Shelah 1975)

MSO($<$) is undecidable over $[0, 1)$.

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Kamp’s theorem again

Some great news:
Kamp’s theorem applies over all domains (rationals need Stavi connectives)

Theorem (Kamp 1968; Gabbay *et al.* 1980)

LTL+Past is as expressive as FO($<$) (over $\mathbb{R}$).

Separation of temporal logics

A temporal logic formula is

- pure past if it is invariant on flows that agree on the past
- pure present if is invariant on flows that agree on the present
- pure future if is invariant on flows that agree on the future

A temporal logic is separable if all its formulas are equivalent to a boolean combination of pure past, present and future formulas.

Lemma
LTL+Past is separable.

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Proof sketch:

$\Rightarrow$: Model-theoretic argument.

$\Leftarrow$: Use separation to eliminate quantified variables.

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Qualitative extensions: Summary

	FO($<$) decidable?	MSO($<$) decidable?	LTL expressively complete?
Classical	Yes	Yes	Yes
Integer time	Yes	Yes	Yes
Rational time	Yes	Yes	Sort of
Real time	Yes	No	Yes

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Motivation

Timed systems are everywhere:

Want to specify:

"If I press the brake pedal then
the pads will be applied within 0.1ms."

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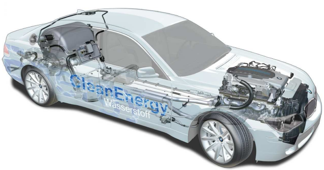

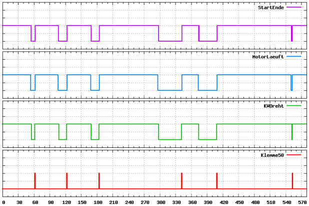

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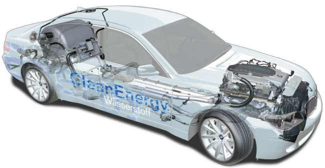

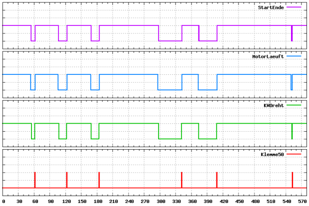

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Motivation

Timed systems also occur in:

- Hardware circuits
- Communication protocols
- Cell phones
- Plant controllers
- Aircraft navigation systems
- …

Example: Timed automata

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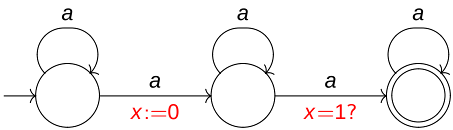

$L(A)$:

Adding time metrics to the models

We want to add a metric to the model so we can enforce certain timing constraints, for example:

"Apply brake pads between 5 *to* 10 *time units after pedal is pushed"*.

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ all have suitable distance metrics.

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$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ all have suitable distance metrics.

Metric predicate logic

We add a unary function $+1$ to the predicate logics to model moving to 1 time unit into the future.

"Apply brake pads between 5 *to* 10 *time units after pedal is pushed"*

becomes

$$\forall x.\mathrm{PEDAL}(x) \rightarrow \exists y.\,(x{+}5 < y < x{+}10) \wedge \mathrm{BRAKE}(y),$$

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Temporal logics: MTL

Metric Temporal Logic (MTL) [Koymans; de Roever; Pnueli ∼1990] is a central quantitative specification formalism for timed systems.

MTL = LTL + timing constraints on operators:

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Formally,

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Decidability

Immediately we have

$$\text{LTL} \subseteq \text{MTL} \subseteq \text{MTL+Past} \subseteq \text{TPTL} \subseteq \text{FO}(<, +1) \subseteq \text{MSO}(<, +1)$$

Over the integers $+1$ is definable in FO($<$):

$$y = x + 1 \equiv y > x \wedge \forall z . z < x \vee z > y$$

So LTL = FO($<$,$+1$) over $\mathbb{N}$ and $\mathbb{Z}$.

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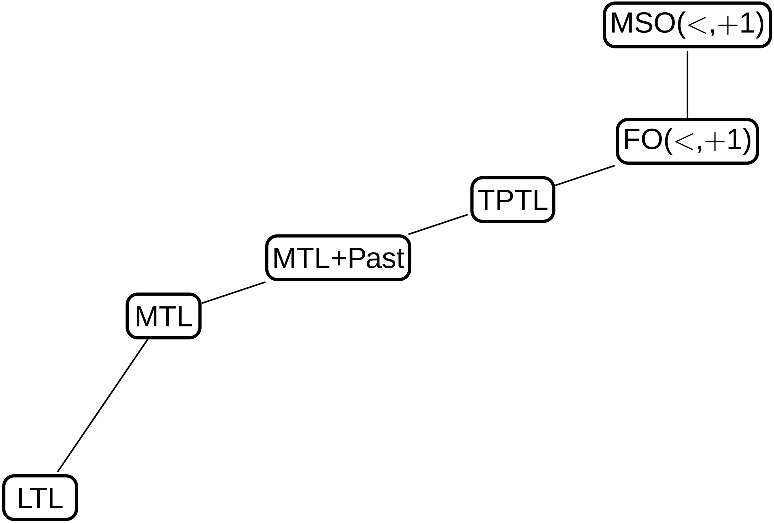

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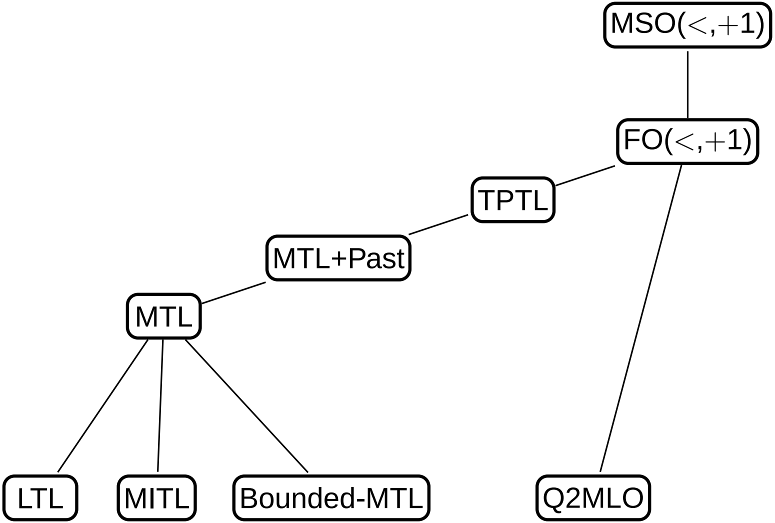

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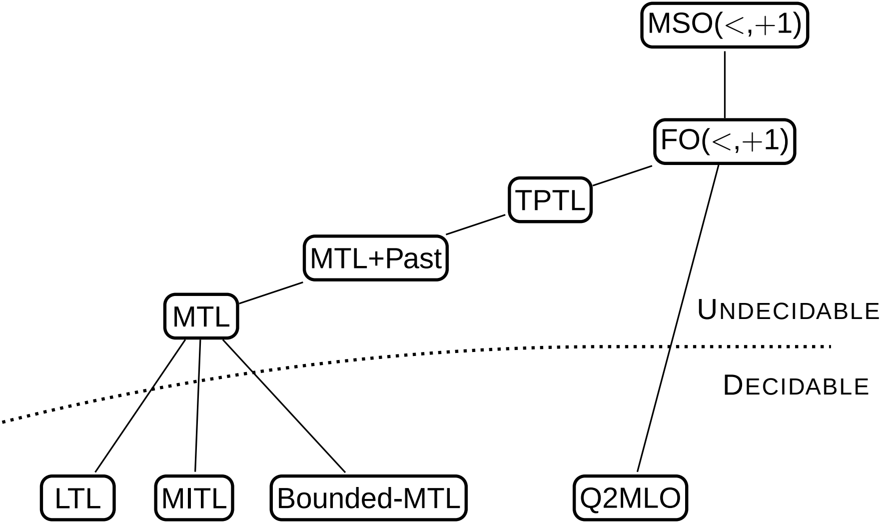

Digression: Expressiveness in bounded time

In reality timed systems often have a “time-out”, where behaviour is not defined after a certain length of time.

We can model this by considering bounded time domains rather than $\mathbb{R}$ or $\mathbb{R}_{\geq 0}$.

Theorem (Ouaknine, Rabinovich & Worrell 2009)

Over bounded time,

- *MSO($<$,$+1$) is decidable, and*
- *MTL is as expressive as FO($<$,$+1$).*

Proof sketch:

- Remove the metric using punctuality,
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"In the next unit time interval there is no b after the last a"

In FO($<$, $+1$):

$$\varphi(z) = \exists x . a(x) \wedge \forall y . (x < y < z + 1) \rightarrow \neg b(y).$$

In MTL+Past:

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MTL+Past is strictly more expressive than MTL.

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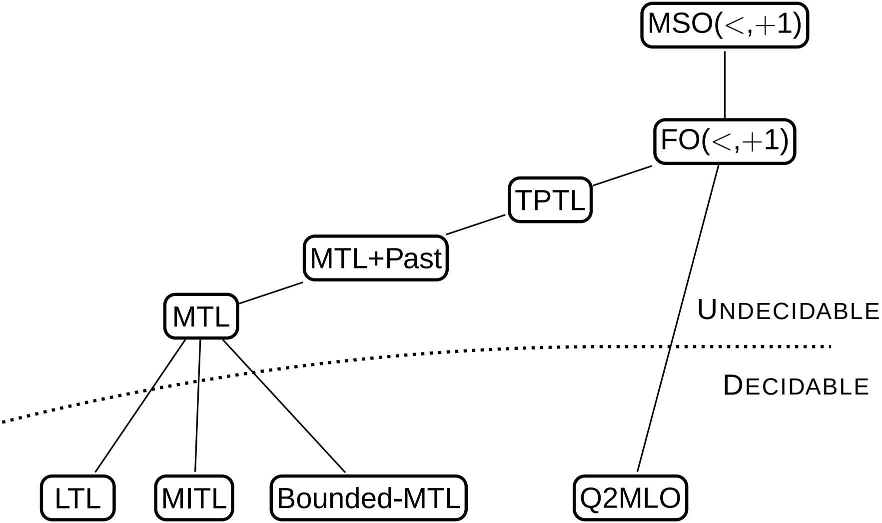

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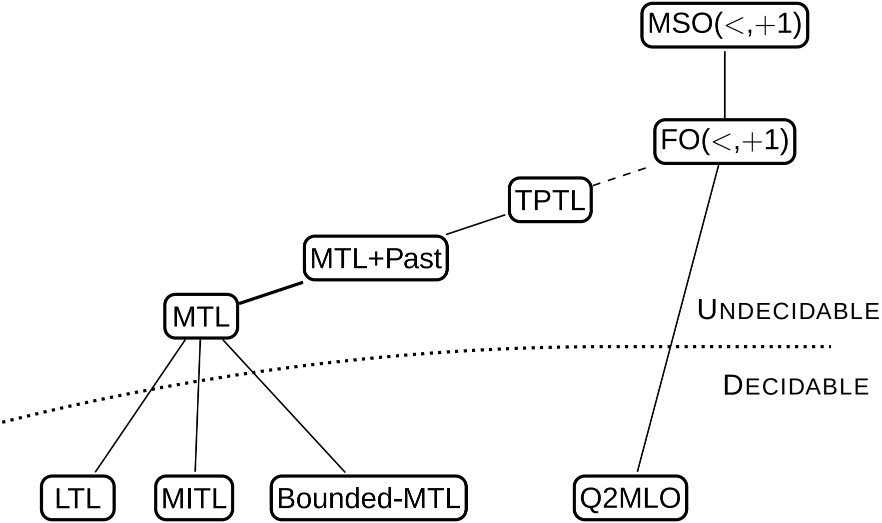

Two brand new results

A metric temporal formula is:

- pure distant past if it is invariant on flows that agree on $(-1, -\infty)$
- pure distant future if it is invariant on flows that agree on $(1, \infty)$
- bounded if there is an N such that it is invariant on all flows that agree on $(-N, N)$

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Theorem (H., Ouaknine, Worrell 2012)

Let $\mathcal{L}$ be a temporal logic.

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