The Expressive Completeness of Metric Temporal Logic

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Highlights of Logic, Games and Automata

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Reasoning about time

LTL has emerged as the definitive temporal logic.

Pros:

- Model-checking in PSPACE
- Human readable
- As expressive as first order logic [Kamp 68]

Con:
LTL cannot express quantitative properties

"*If I press the brake pedal then the pads will be applied.*"

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Metric Temporal Logic (MTL)

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[Koymans; de Roever; Pnueli ∼1990] is LTL with timing constraints added to the temporal modalities

$$\begin{array}{lll} \theta & ::= & P \mid \theta_1 \wedge \theta_2 \mid \theta_1 \vee \theta_2 \mid \neg\theta \\ & & \mid \theta_1 \, \mathbf{U} \, \theta_2 \quad (\theta_2 \text{ holds and Until then } \theta_1 \text{ holds}) \\ & & \mid \theta_1 \, \mathbf{S} \, \theta_2 \quad (\theta_2 \text{ held and } \theta_1 \text{ has held Since}) \end{array}$$

where I is an interval with endpoints taken from some $\mathcal{K} \subseteq \mathbb{R}$.

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Predicate Logic

FO($<$): First-order logic with linear order $<$ and monadic predicates P.

FO$_{\mathcal{K}}$: Given a set $\mathcal{K} \subseteq \mathbb{R}$ of constants we add many unary functions $\{+c : c \in \mathcal{K}\}$ to FO($<$) to model moving c time units into the future.

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$$\texttt{PEDAL}(x) \to \exists y.\, (x < y < x+1) \wedge \texttt{BRAKE}(y),$$

How expressive is MTL?

$$\mathrm{MTL}_{\mathcal{K}} \subseteq \mathrm{FO}_{\mathcal{K}}$$

When is there equality?

- With no constants: $\mathrm{MTL}_{\{0\}} = \mathrm{FO}_{\{0\}}$ [Kamp 68]
- With integer constants: $\mathrm{MTL}_{\mathbb{Z}} \neq \mathrm{FO}_{\mathbb{Z}}$ [Hirshfeld and Rabinovich 07]
- With rational constants: $\mathrm{MTL}_{\mathbb{Q}} = \mathrm{FO}_{\mathbb{Q}}$ [H., Ouaknine and Worrell 13]

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Failure of Kamp's theorem

$MTL_{\mathbb{Z}}$ is unable to express:

"*P occurs twice in the next time interval.*"

$$\begin{aligned}\varphi \quad = \quad & \mathbf{F}_{(0,)}\left(P \wedge \mathbf{F}_{(0,)} P\right) \quad \vee \\ & \left(\mathbf{F}_{(0,)} P \wedge \mathbf{F}_{(,)} P\right) \quad \vee \\ & \mathbf{F}_{=}\left(\mathbf{P}_{(0,)}\left(P \wedge \mathbf{P}_{(0,)} P\right)\right)\end{aligned}$$

Corollary

"P occurs twice in the next time interval" is expressible in $MTL_{\mathbb{Q}}$.

Failure of Kamp's theorem

$\mathrm{MTL}_{\mathbb{Z}}$ is able to express:

"*P occurs twice in the next two time intervals.*"

$$\begin{aligned}\varphi \quad = \quad & \mathbf{F}_{(0,1)}\big(\mathtt{P} \wedge \mathbf{F}_{(0,1)}\mathtt{P}\big) \quad \vee \\ & (\mathbf{F}_{(0,1)}\mathtt{P} \wedge \mathbf{F}_{(1,2)}\mathtt{P}) \quad \vee \\ & \mathbf{F}_{=2}\Big(\mathbf{P}_{(0,1)}\big(\mathtt{P} \wedge \mathbf{P}_{(0,1)}\mathtt{P}\big)\Big)\end{aligned}$$

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Failure of Kamp's theorem

$\mathrm{MTL}_{\mathbb{Z}}$ is able to express:

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$$\begin{aligned}\varphi \quad = \quad & \mathbf{F}_{(0,\frac{1}{2})}\big(\mathtt{P} \wedge \mathbf{F}_{(0,\frac{1}{2})}\mathtt{P}\big) \quad \vee \\ & \big(\mathbf{F}_{(0,\frac{1}{2})}\mathtt{P} \wedge \mathbf{F}_{(\frac{1}{2},1)}\mathtt{P}\big) \quad \vee \\ & \mathbf{F}_{=1}\Big(\mathbf{P}_{(0,\frac{1}{2})}\big(\mathtt{P} \wedge \mathbf{P}_{(0,\frac{1}{2})}\mathtt{P}\big)\Big)\end{aligned}$$

Corollary

"P occurs twice in the next time interval" is expressible in $\mathrm{MTL}_{\mathbb{Q}}$.

Counting is all you need...

Theorem (H. 13)

$MTL_{\mathbb{Z}}$ with counting modalities has the same expressive power as $FO_{\mathbb{Z}}$.

Observation/Exercise

1. If $\mathcal{K}$ is a non-dense subgroup of $\mathbb{R}$ then $\mathcal{K} = \epsilon \cdot \mathbb{Z}$.
2. If $\mathcal{K}$ is a non-trivial dense subgroup of $\mathbb{R}$ then $MTL_{\mathcal{K}}$ can express the counting modalities.

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When is MTL expressively complete?

Theorem (H. 13)

$MTL_{\mathcal{K}} = FO_{\mathcal{K}}$ if and only if $\mathcal{K}$ is dense.