Reachability in 2-clock automata:
A deceptively hard problem

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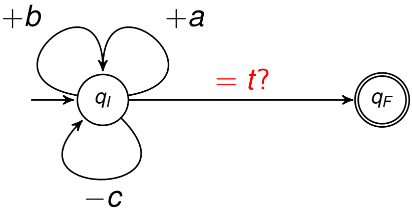
Université Libre de Bruxelles

MFV, December 2013

The problem

Bounded one-counter machine:



- One counter, taking integer values in $[0, M)$
- One guarded transition $q_I \to q_F$
- Three increment/decrement transitions $q_I \to q_I$

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Given $a, b, c, M, t \in \mathbb{N}$ can the machine reach q_F?

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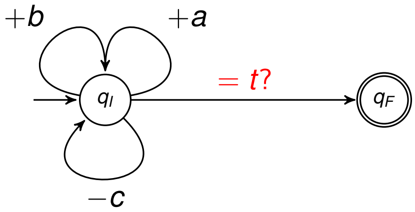

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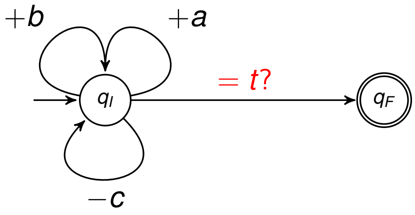

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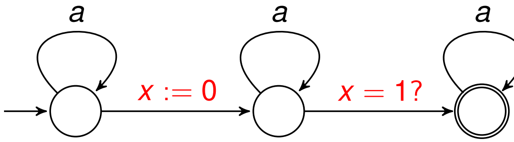
Motivation: Timed automata

Timed automata were introduced by Rajeev Alur at Stanford during his PhD thesis under David Dill.



Accepts timed words over $\{a\}$ where there are two a's exactly one time unit apart

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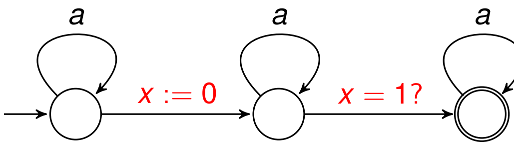

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Many problems undecidable, but what about reachability?

Reachability in Timed Automata

PSPACE-complete with $2n + 1$ clocks [AD90]

PSPACE-complete with 3 clocks [CY92]

NL-complete with 1 clock [LMS04]

Two-clock reachability is equivalent to
bounded one-counter reachability [HOW12]

NP-complete for unbounded one-counter reachability [HKOW09]

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From timed automata to counter machines

Idea: Store difference of two clocks in counter value

Problem: How to do inequalities?

Solution: Impose upper-bound limit on counter value!

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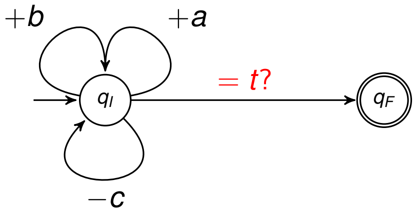

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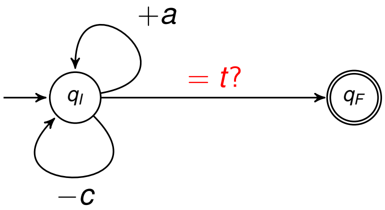

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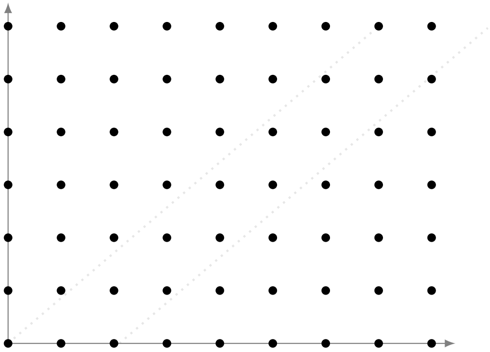
A geometric interpretation

Run of the machine can be thought of as a 2-dimensional walk:



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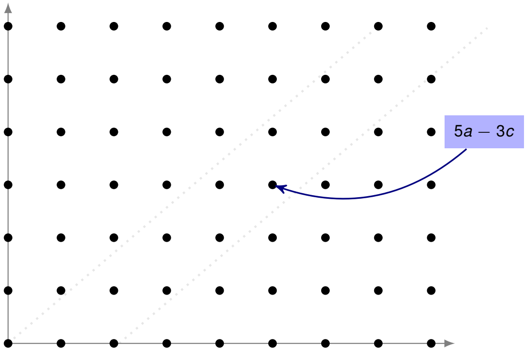

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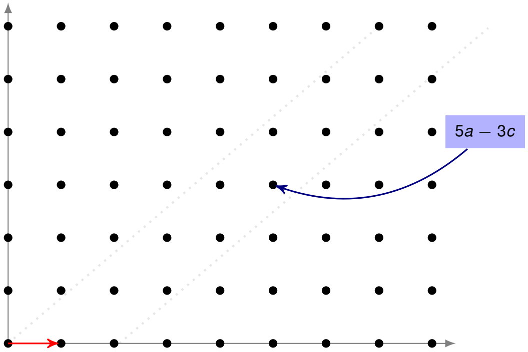

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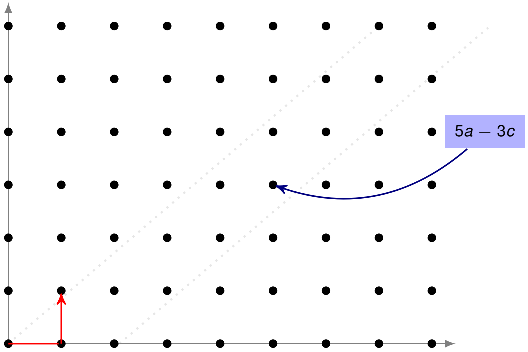

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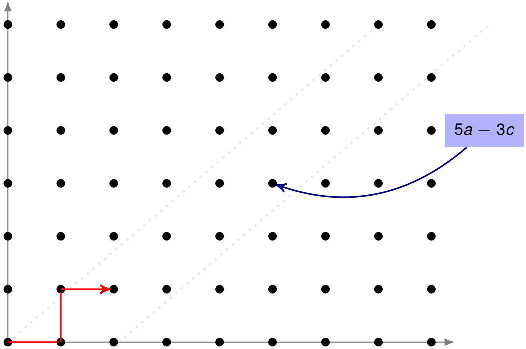

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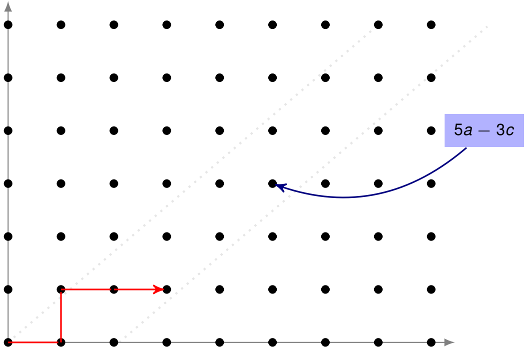

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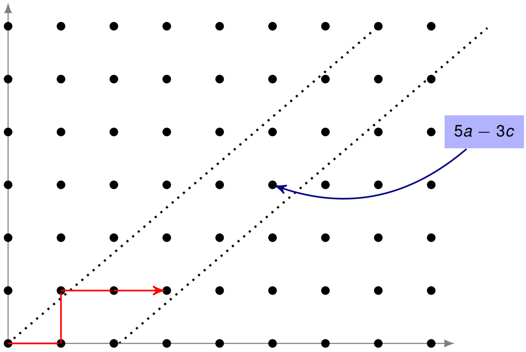

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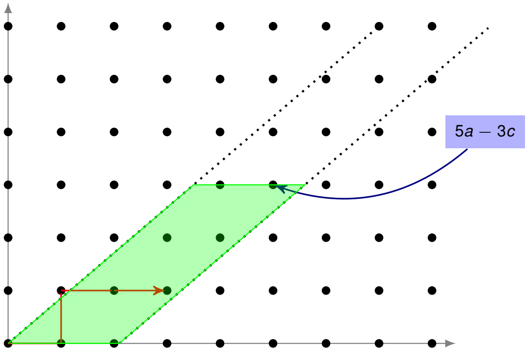

Solving the 2-D case

Theorem

Let (m, n) *be a feasible point, and let* P *be the parallelogram bounded by the parallel lines,* $x = 0$ *and* $x = m$*. Then there is a walk from* $(0, 0)$ *to* (m, n) *if and only if* P *contains at least* $m + n + 1$ *lattice points.*

Theorem (Pick's theorem)

Let P *be a convex polyhedron with vertices on lattice points. Then*

$$Area(P) = \#interior\ points + \frac{1}{2}\#boundary\ points.$$

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There are analogues of Pick's theorem for non-lattice vertices and in more than 2 dimensions.

Unfortunately there is no analogue of the first theorem in 3 dimensions.

A graph theoretic perspective

Consider the configuration graph $G^M_{a,b,c}$ of the counter machine:

- Vertices are integers in $[0, M)$
- a-edges from n to $n+a$; b-edges from n to $n+b$ and c-edges from n to $n-c$.

Reachability in counter machine = Reachability in $G^M_{a,b,c}$.

$G^M_{a,b,c}$ has nice properties:

- $G^M_{a,c}$ is a subgraph of $G^M_{a,b,c}$
- $G^{M'}_{a,b,c}$ is a subgraph of $G^M_{a,b,c}$ if $M' \leq M$.

What does $G^M_{a,b,c}/G^M_{a,c}$ look like?

Some group theory

Given a group G and a set $S \subseteq G$ the Cayley graph of G with respect to S is the graph with

- Vertices are elements of G generated by S
- There is an (s-)edge from x to y if $y = x \cdot s$ for some $s \in S$.

$G^M_{a,b,c}$ is an induced subgraph of the Cayley graph of $(\mathbb{Z}, +)$ with respect to $\{a, b, -c\}$!

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What does $G_{a,c}^M$ look like?

Lemma

- *If $M \geq a + c$ then every vertex has out-degree at least 1 and in-degree at least 1.*
- *If $M \leq a + c$ then every vertex has out-degree at most 1 and in-degree at most 1.*

Corollary

If $M = a + c$ then $G_{a,c}^M$ is a set of ($\gcd(a, c)$) disjoint cycles.

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If $M \geq a + c$ and $\gcd(a, c) | t$ then there is a path from 0 to t.

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From 3-D to 2-D

Theorem

If $M \geq a + c$ then reachability in $G^M_{a,b,c}$ reduces to reachability in $G^d_{b,d-b}$ where $d = \mathrm{gcd}(a, c)$.

What about if $M < a + c$?

$G^M_{a,b,c}$ is a set of disjoint paths. How to tell if s and t are on the same path?

Solution: Look at the maximum value between s and t on $G^{a+c}_{a,c}$.

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A modular arithmetic perspective

The vertices of $G_{a,c}^{a+c}$ are $[0, a+c)$ which are the integers modulo $a+c$. Also, $+a \equiv -c \pmod{a+c}$.

Traversing $G_{a,c}^{a+c}$ is equivalent to taking multiples of a modulo $a+c$.

Problem

Given a, M, t let n be the smallest positive integer such that $t \equiv n \cdot a \pmod{M}$. What is the maximum value of $\{i \cdot a \pmod{M} \;:\; 0 \leq i \leq n\}$?

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Fibonacci representation

Every natural number can be written as a sum of Fibonacci numbers,

$$n = \sum_{i=1}^{k} \delta_i F_i$$

where $\delta_i \in \{0, 1\}$ and F_i is the i-th Fibonacci number. With the rewrite rule $011 \rightarrow 100$ this representation is unique. This is the Fibonacci representation.

Facts about the Fibonacci representation

- The Fibonacci representation of n is logarithmic in the size of n
- There is a 1-1 correspondence with fit-strings and polynomials in $\mathbb{Z}[X]/(X^2 - X - 1)$
- There is a 1-1 correspondence with fit-strings and elements of $\mathbb{Z}(\varphi)$
- The Fibonacci representation can be seen as the "base-φ representation".

Negafibonacci representation

Every integer can be written as a sum of negaFibonacci numbers,

$$n = \sum_{i=1}^{k} \delta_i F_i$$

where $\delta_i \in \{0, 1\}$ and F_i is the $(-i)$-th Fibonacci number.

Application: Navigating a tiling of the hyperbolic plane [Knuth].

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Euclidean representation

Let $r_0 = a + c$, $r_1 = a$ and consider the sequence of r_i and q_i generated by the Euclidean algorithm via

$$r_i = q_{i+1} \cdot r_{i+1} + r_{i+2}.$$

Theorem

Every integer $N \in [-a, c)$ has a unique representation of the form

$$N = \sum_{i=1}^{m} (-1)^{i+1} b_i \cdot r_i$$

where $0 \leq b_1 \leq q_1 - 1$; $0 \leq b_k \leq q_k$, for $k \geq 2$ and $b_k = 0$ if $b_{k+1} = q_{k+1}$. Moreover, the difference between lexicographic neighbours in this encoding is either a or $-c$.

Euclidean representation example

Consider $a = 17$, $c = 5$:

$$
\begin{array}{rcll}
22 & = & 1.17 + 5 & (q_1 = 1) \\
17 & = & 3.5 + 2 & (q_2 = 3) \\
5 & = & 2.2 + 1 & (q_3 = 2) \\
2 & = & 2.1 + 0 & (q_4 = 2)
\end{array}
$$

Permissible $b_4 b_3 b_2$ ($b_1 = 0$):

000(0)	010(2)	020(4)	103(−16)	113(−14)	202(−12
001(−5)	011(−3)	100(−1)	110(1)	120(3)	203(−17
002(−10)	012(−8)	101(−6)	111(−4)	200(−2)	
003(−15)	013(−13)	102(−11)	112(−9)	201(−7)	

Algorithm for 2-D reachability

Finding the maximum value between s and t on $G_{a,c}^{a+c}$ then becomes:

- Compute the representation of s and t
- Solve the resulting linear constraint problem to find the maximum value between s and t

Ostrowski representation

The Ostrowski representation can be seen as a generalization of (nega)Fibonacci representation. Given $\alpha \in \mathbb{R}_{\geq 0}$ let

$[a_0, a_1, \ldots]$ be the continued fraction representation of α. That is:

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}}$$

Let $\frac{p_n}{q_n}$ represent the n-th approximation of α and let $\theta_n = q_n\alpha - p_n$.

Ostrowski representation

Theorem

If α is irrational then

- *Every natural number N can be written uniquely in the form*

$$N = \sum_{i=1}^{m} b_i q_{i-1}$$

  where $0 \leq b_1 \leq a_1 - 1$; $0 \leq b_k \leq a_k$, *for* $k \geq 2$ *and* $b_k = 0$ *if* $b_{k+1} = a_{k+1}$.

- *Every real number* $x \in [-\alpha, 1-\alpha)$ *can be written uniquely in the form*

$$x = \sum_{i=1}^{\infty} b_i \theta_{i-1}$$

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What's going on?

Intuitively $\sum_{i=1}^{\infty} b_i \theta_{i-1}$ is the fractional part (shifted to $[-\alpha, 1-\alpha)$) of $N\alpha$ where

$$N = \sum_{i=1}^{\infty} b_i q_{i-1}.$$

Integer multiples of $\frac{a}{a+c}$ modulo 1 are equivalent to integer multiples of a modulo $a + c$

When α is rational,

- The continued fraction for α is finite so the Ostrowski representation is finite, and
- $\theta_n = (-1)^{n+1} \frac{r_n}{r_0}$ where r_i is derived from the Euclidean algorithm.

Corollary

The Euclidean representation is equivalent to the Ostrowski representation

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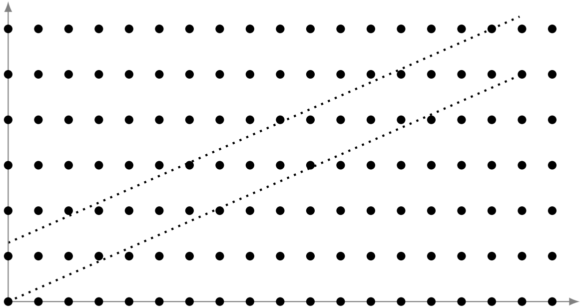
Returning to geometry



001 010 100 000

$(1,0)$ $(3,1)$ $(7,2)$ $(17,5)$

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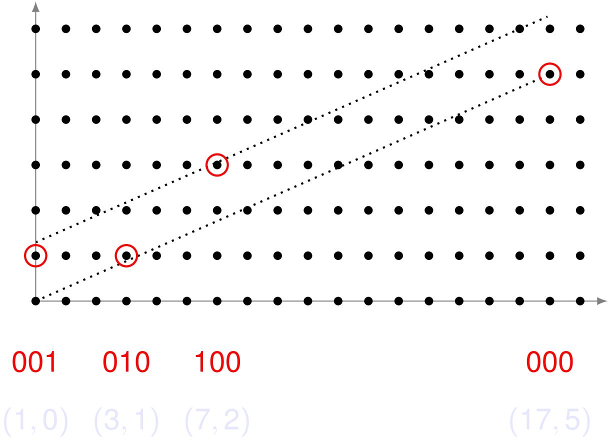

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