The Expressiveness of Metric Temporal Logic II:

This time it's irrational!

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(Joint work with Joël Ouaknine and James Worrell)

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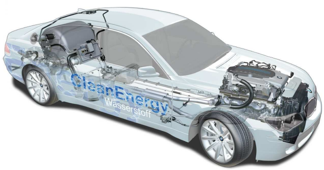
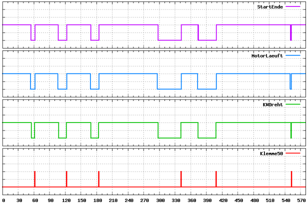
Reasoning about time

Timed systems are everywhere:

- Hardware circuits
- Communication protocols
- Cell phones
- Plant controllers
- Aircraft navigation systems
- ...

Reasoning about time





Want to specify:

“*If I press the brake pedal then the pads will be applied*.”

“*If the brakes are applied then the pedal has been pressed*.”

Expressiveness *vs* **Computability**

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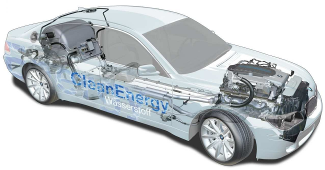

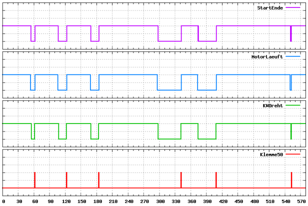

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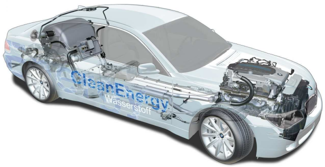

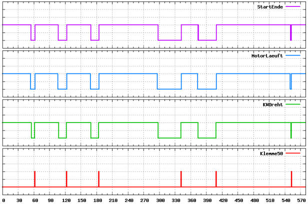

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Expressiveness *vs* **Computability**

Classic temporal logic

Metric temporal logic

Extending Kamp’s Theorem (again)

Temporal models

- A set **MP** of propositions: $P, Q, R, \ldots$
- Continuous time model: $\mathbb{R}$

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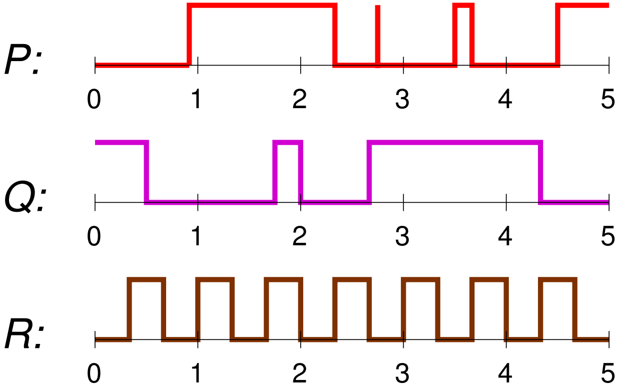

Classic temporal predicate logic

FO($<$): First-order logic with $<$ and monadic predicates for each proposition $P \in \mathbf{MP}$:

$$\varphi ::= x < y \mid P(x) \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi \mid \forall x\, \varphi \mid \exists x \varphi$$

For example:

$$\forall x\, .\, \mathrm{PEDAL}(x) \rightarrow \exists y\, .\, ((y > x) \wedge \mathrm{BRAKE}(y)).$$

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Linear Temporal Logic (LTL): Propositional logic with temporal modalities:

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LTL is all you need

LTL has emerged as *the* definitive temporal logic in the classical setting.

Theorem (Kamp 1968, GPSS 1980)

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In reality, timed systems are usually quantitative.

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In reality, timed systems are usually quantitative.

Want to specify:

"*If I press the brake pedal then*
the pads will be applied between 0.5ms and 1ms."

Adding time metrics to the models

We want to add a metric to the model so we can enforce certain timing constraints, for example:

"Apply brake pads between 5 *to* 10 *time units after pedal is pushed"*.

$\mathbb{R}$ has a distance metric $\rightsquigarrow$ use real intervals for timing constraints.

- Traditional approach: intervals over $\mathbb{Z}$.
- Continuous but finitely presentable: intervals over $\mathbb{Q}$ (seen in Part I).
- **NEW!** Intervals over an arbitrary additive subgroup of $\mathbb{R}$...

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Additive subgroup?

- Can easily form integer linear combinations of timing constants.
- Integer linear combinations of $\mathcal{K}$ = Subgroup of $(\mathbb{R}, +)$ generated by $\mathcal{K}$.

Motivation

- Includes most general case ($\mathcal{K} = \mathbb{R}$)
- Generalizes previous cases ($\mathcal{K} = \mathbb{Z},\ \mathbb{Q},$ or $\{0\}$)
- Can be used to model multiple independent asynchronous timing systems (e.g. $\mathbb{Z}[\sqrt{2}]$)

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Metric Predicate Logic

We add many unary functions $\{+c : c \in \mathcal{K}\}$ to FO($<$) to model moving c time units into the future.

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a formula of $\text{FO}_{\{5,10\}}$.

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Metric Temporal Logic

Metric Temporal Logic (MTL) [Koymans; de Roever; Pnueli ∼1990] is a central quantitative specification formalism for timed systems.

$\text{MTL}_{\mathcal{K}}$ = LTL + timing constraints on operators:

$$\mathbf{G}\,(\text{PEDAL} \rightarrow \mathbf{F}_{(5,10)}\,\text{BRAKE})$$

Formally,

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Extending Kamp’s Theorem (again)

Kamp's theorem restated

Theorem (Kamp 1968)

$MTL_{\{0\}}$ has the same expressive power as $FO_{\{0\}}$.

Part I recap

Theorem (Hirshfeld & Rabinovich 2007)

$MTL_{\mathbb{Z}}$ is strictly less expressive than $FO_{\mathbb{Z}}$.

Part I recap

Theorem (H., Ouaknine & Worrell 2013)

$MTL_{\mathbb{Q}}$ has the same expressive power as $FO_{\mathbb{Q}}$.

What about $MTL_{\mathbb{R}}$? or $MTL_{\mathbb{Z}[\sqrt{2}]}$?

A true extension of Kamp's theorem

Theorem (H. 2013)

$MTL_{\mathcal{K}}$ = $FO_{\mathcal{K}}$ if and only if $\mathcal{K}$ is dense.

Proof: “Only if”

Lemma

If $\mathcal{K}$ is a non-dense additive subgroup of $\mathbb{R}$ then $\mathcal{K} = \epsilon\mathbb{Z}$ for some $\epsilon \in \mathbb{R}$.

Proof: “If”

Recall proof in Part I:

1. Use metric separation to reduce to bounded formulas.
2. Scale $FO_{\mathbb{Q}}$ formula to get a formula in $FO_{\mathbb{Z}}$.
3. Use “stacking” to remove the $+1$ function.
4. Use denseness of $\mathbb{Q}$ to express LTL statements restricted to a single time interval.
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Proof: “If”

1. Use more general metric separation to reduce to bounded formulas.
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Separation of temporal logics

A temporal logic formula is

- pure past if it is invariant on flows that agree on the past
- pure present if is invariant on flows that agree on the present
- pure future if is invariant on flows that agree on the future

A temporal logic is separable if all its formulas are equivalent to a boolean combination of pure past, present and future formulas.

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Lemma

LTL is separable.

Gabbay's theorem

Theorem (Gabbay 1981)

A temporal logic is expressively complete if and only if it is separable.

Quantitative separation

Separation does not hold in the quantitative setting.

For example,

$$\mathbf{G}(\mathsf{BRAKE} \rightarrow \mathbf{P}_{(5,10)}\mathsf{PEDAL})$$

General quantitative separation

Given a constant $c > 0$, a metric temporal formula is:

- pure c-distant past if it is invariant on flows that agree on $(-\infty, -c)$
- pure c-distant future if it is invariant on flows that agree on (c, ∞)
- bounded if there is an N such that it is invariant on all flows that agree on $(-N, N)$

A temporal logic with constants from $\mathcal{K}$ is generally metrically separable if every formula is equivalent, for some $c \in \mathcal{K}_{>0}$, to a boolean combination of pure c-distant past, pure c-distant future and bounded formulas.

Lemma

$MTL_{\mathcal{K}}$ is generally metrically separable for non-trivial $\mathcal{K}$.

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- bounded if there is an N such that it is invariant on all flows that agree on $(-N, N)$

A temporal logic with constants from $\mathcal{K}$ is generally metrically separable if every formula is equivalent, for some $c \in \mathcal{K}_{>0}$, to a boolean combination of pure c-distant past, pure c-distant future and bounded formulas.

Lemma

$MTL_{\mathcal{K}}$ is generally metrically separable for non-trivial $\mathcal{K}$.

General quantitative separation

Given a constant $c > 0$, a metric temporal formula is:

- pure c-distant past if it is invariant on flows that agree on $(-\infty, -c)$
- pure c-distant future if it is invariant on flows that agree on (c, ∞)
- bounded if there is an N such that it is invariant on all flows that agree on $(-N, N)$

A temporal logic with constants from $\mathcal{K}$ is generally metrically separable if every formula is equivalent, for some $c \in \mathcal{K}_{>0}$, to a boolean combination of pure c-distant past, pure c-distant future and bounded formulas.

Corollary

$MTL_{\mathcal{K}} = FO_{\mathcal{K}}$ iff $MTL_{\mathcal{K}}$ can express all bounded $FO_{\mathcal{K}}$ formulas.

Proof: “If”

1. Use more general metric separation to reduce to bounded formulas.
2. Use a normal form for $FO_{\mathcal{K}}$ formulas to remove $+c$ functions.
3. Use denseness of $\mathcal{K}$ to express LTL statements restricted to an interval.

Removing the unary functions

First move the unary functions to the free variable (removing from predicates as before).

$$
\begin{aligned}
\varphi(x) &= \exists y \in (x, x+1)\ \exists z \in (y, y+\sqrt{2}) \ldots \\
&= \exists y \in (x, x+1)\ \big(\exists z \in (y, x+1) \ldots \\
&\qquad \vee \exists z \in (x+1, y+\sqrt{2}) \ldots \big) \\
&= \exists y \in (x, x+1)\ \big(\exists z \in (y, x+1) \ldots \\
&\qquad \vee \exists z' \in (x+1-\sqrt{2}, y) \ldots \big) \\
\varphi(x_0, x_1, x_2) &= \exists y \in (x_1, x_2)\ \big(\exists z \in (y, x_2) \ldots \vee \exists z' \in (x_0, y) \ldots \big)
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Removing the unary functions

Replace the "milestones" ($\{x+1-\sqrt{2}, x, x+1\}$) with new variables to obtain a FO($<$) formula.

$$
\begin{aligned}
\varphi(x) &= \exists y \in (x, x+1)\, \exists z \in (y, y+\sqrt{2}) \dots \\
&= \exists y \in (x, x+1)\, \big(\exists z \in (y, x+1) \dots \\
&\qquad\qquad \vee\, \exists z \in (x+1, y+\sqrt{2}) \dots \big) \\
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\end{aligned}
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Corollary

$MTL_{\mathcal{K}}$ = $FO_{\mathcal{K}}$ iff $MTL_{\mathcal{K}}$ can express all bounded $FO_{\{0\}}$ formulas.

Removing the unary functions

Use a model-theoretic argument to break this into formulas on the intervals $\{x_0\}, (x_0, x_1), \{x_1\}, \ldots$

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Failure of Kamp's theorem

$MTL_{\mathbb{Z}}$ is unable to express:

"*P occurs twice in the next time interval.*"

In $FO_{\mathbb{Z}}$:

$$\varphi(z) = \exists x.\exists y.(z < x < z +) \wedge (z < y < z +) \wedge \mathrm{P}(x) \wedge \mathrm{P}(y).$$

In $MTL_{\mathbb{Z}}$:

$$\left(\mathbf{F}_{(0,1)}\mathrm{P} \wedge \mathbf{F}_{(1,2)}\mathrm{P}\right) \quad \vee$$
$$\mathbf{F}_{=2}\left(\mathbf{P}_{(0,1)}\left(\mathrm{P} \wedge \mathbf{P}_{(0,1)}\mathrm{P}\right)\right)$$

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???

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Failure of Kamp's theorem

$\mathsf{MTL}_{\mathbb{Z}}$ is able to express:

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In $\mathsf{MTL}_{\mathbb{Z}}$:

$$\begin{aligned} \varphi \quad = \quad & \mathbf{F}_{(0,1)}\big(\mathsf{P} \wedge \mathbf{F}_{(0,1)}\mathsf{P}\big) \quad \vee \\ & \big(\mathbf{F}_{(0,1)}\mathsf{P} \wedge \mathbf{F}_{(1,2)}\mathsf{P}\big) \quad \vee \\ & \mathbf{F}_{=2}\Big(\mathbf{P}_{(0,1)}\big(\mathsf{P} \wedge \mathbf{P}_{(0,1)}\mathsf{P}\big)\Big) \end{aligned}$$

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$$\begin{aligned} \varphi \quad = \quad & \mathbf{F}_{(0,1)}\big(P \wedge \mathbf{F}_{(0,1)}P\big) \quad \vee \\ & \big(\mathbf{F}_{(0,1)}P \wedge \mathbf{F}_{(1,2)}P\big) \quad \vee \\ & \mathbf{F}_{=2}\Big(\mathbf{P}_{(0,1)}\big(P \wedge \mathbf{P}_{(0,1)}P\big)\Big) \end{aligned}$$

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Adding granularity

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Corollary

"P occurs twice in the next time interval" is expressible in $\text{MTL}_{\mathbb{Q}}$.

In fact, $\text{MTL}_{\mathcal{K}}$ can express any LTL (and hence $\text{FO}_{\{0\}}$) formula "in the next time interval" as long as $\mathcal{K}$ is dense and non-trivial.

Corollary

$\text{MTL}_{\mathcal{K}}$ can express $\text{FO}_{\mathcal{K}}$ iff $\mathcal{K}$ is dense and non-trivial.

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$$\begin{aligned}\varphi \quad = \quad & \mathbf{F}_{(0,\frac{1}{2})}\ \big(\mathsf{P} \wedge \mathbf{F}_{(0,\frac{1}{2})}\ \mathsf{P}\big) \quad \vee \\ & \big(\mathbf{F}_{(0,\frac{1}{2})}\ \mathsf{P} \wedge \mathbf{F}_{(\frac{1}{2},1)}\ \mathsf{P}\big) \quad \vee \\ & \mathbf{F}_{=1}\Big(\mathbf{P}_{(0,\frac{1}{2})}\ \big(\mathsf{P} \wedge \mathbf{P}_{(0,\frac{1}{2})}\ \mathsf{P}\big)\Big)\end{aligned}$$

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Adding granularity

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Corollary

"P occurs twice in the next time interval" is expressible in MTL$_{\mathbb{Q}}$.

In fact, MTL$_{\mathcal{K}}$ can express any LTL (and hence FO$_{\{0\}}$) formula "in the next time interval" as long as $\mathcal{K}$ is dense and non-trivial.

Corollary

MTL$_{\mathcal{K}}$ can express FO$_{\mathcal{K}}$ iff $\mathcal{K}$ is dense and non-trivial.

A true extension of Kamp's theorem

Theorem

$MTL_{\mathcal{K}} = FO_{\mathcal{K}}$ if and only if $\mathcal{K}$ is dense.

The Expressive Completeness of Metric Temporal Logic II$\frac{1}{2}$:

Count on this

Counting modalities

The counting modalities $\{\mathbf{C}_n : n \in \mathbb{N}\}$ were introduced by Hirshfeld & Rabinovich in 2007.

Intuitively, $\mathbf{C}_n\varphi$ asserts that φ holds in at least n distinct points in the next unit time interval.

MTL with counting is $\text{MTL}_{\mathbb{Z}}$ with the addition of the counting modalities.

A decidability result

Hirshfeld & Rabinovich considered MITL with counting (MTL with counting without singleton intervals).

Theorem (Hirshfeld & Rabinovich 2007)

MITL with counting is decidable.

An expressiveness result

Adding punctuality to MITL with counting gives it the power to express every bounded $FO_{\mathbb{Z}}$ formula, and hence every $FO_{\mathbb{Z}}$ formula.

Theorem (H. 2013)

MTL with counting is expressively equivalent to $FO_{\mathbb{Z}}$.

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Conclusions and further work

- Precisely characterized when MTL has the same expressive power as first-order logic.
- Adding counting to the non-equivalent cases gives full expressive power.

Still to do:

- Cost of expressibility.
- Generalization of Gabbay's Theorem.
- Extension to more expressive metric temporal logics.

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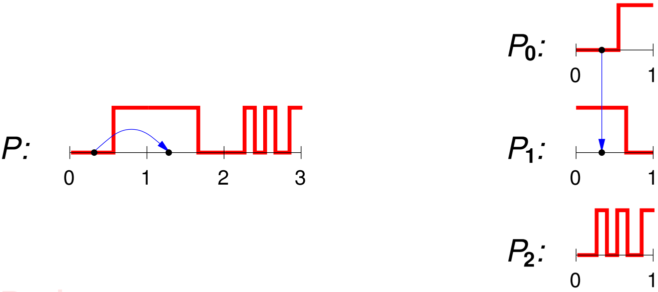
From FO(<,+1) to FO(<)



Replace every:

- $\forall x\, \psi(x)$ by $\forall x\, (\psi(x) \wedge \psi(x+1) \wedge \psi(x+2))$
- $x + k_1 < y + k_2$ by $\begin{cases} x < y & \text{if } k_1 = k_2 \\ \textbf{true} & \text{if } k_1 < k_2 \\ \textbf{false} & \text{if } k_1 > k_2 \end{cases}$
- $P(x+k)$ by $P_k(x)$
- After converting to MTL, replace P_k with $\mathbf{F}_{=k} P$

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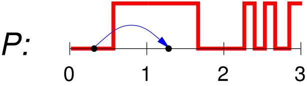

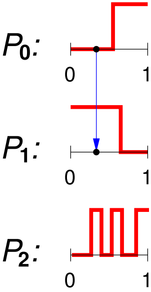

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